# Chapter 5 – Initial-Value Problems for Ordinary Differential Equations

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## Lipschitz Condition and Convexity

**Definition**

A function $f(t, y)$ is said to satisfy a *Lipschitz condition* in the variable $y$ on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists with

$$|f(t, y_1) - f(t, y_2)| \le L|y_1 - y_2|,$$

whenever $(t, y_1), (t, y_2) \in D$. The constant $L$ is called a *Lipschitz constant* for $f$.

**Definition**

A set $D \subset \mathbb{R}^2$ is said to be *convex* if whenever $(t_1, y_1)$ and $(t_2, y_2)$ belong to $D$ and $\lambda$ is in $[0, 1]$, the point $((1 - \lambda)t_1 + \lambda t_2, (1 - \lambda)y_1 + \lambda y_2)$ also belongs to $D$.

## Existence and Uniqueness

**Theorem**

*Suppose $f(t, y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L > 0$ exists with*

$$\left|\frac{\partial f}{\partial y}(t, y)\right| \le L, \quad \text{for all } (t, y) \in D,$$

*then $f$ satisfies a Lipschitz condition on $D$ in the variable $y$ with Lipschitz constant $L$.*

**Theorem**

*Suppose that $D = \{(t, y) \mid a \le t \le b, -\infty < y < \infty\}$ and that $f(t, y)$ is continuous on $D$. If $f$ satisfies a Lipschitz condition on $D$ in the variable $y$, then the initial-value problem*

$$y'(t) = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

*has a unique solution $y(t)$ for $a \le t \le b$.*

## Well-Posedness

**Definition**

The initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

is said to be a *well-posed problem* if:

- A unique solution, $y(t)$, to the problem exists, and
- There exist constants $\varepsilon_0 > 0$ and $k > 0$ such that for any $\varepsilon$, with $\varepsilon_0 > \varepsilon > 0$, whenever $\delta(t)$ is continuous with $|\delta(t)| < \varepsilon$ for all $t$ in $[a, b]$, and when $|\delta_0| < \varepsilon$, the initial-value problem

  $$\frac{dz}{dt} = f(t, z) + \delta(t), \quad a \le t \le b, \quad z(a) = \alpha + \delta_0,$$

  has a unique solution $z(t)$ that satisfies

  $$|z(t) - y(t)| < k\varepsilon \quad \text{for all } t \text{ in } [a, b].$$

## Well-Posedness

**Theorem**

*Suppose $D = \{(t, y) \mid a \le t \le b$ and $-\infty < y < \infty\}$. If $f$ is continuous and satisfies a Lipschitz condition in the variable $y$ on the set $D$, then the initial-value problem*

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

*is well-posed.*

## Euler's Method

Suppose a well-posed initial-value problem is given:

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

Distribute mesh points equally throughout $[a, b]$:

$$t_i = a + ih, \quad \text{for each } i = 0, 1, 2, \dots, N.$$

The *step size* $h = (b - a)/N = t_{i+1} - t_i$.

## Euler's Method

Use Taylor's Theorem for $y(t)$:

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2} y''(\xi_i)$$

for $\xi_i \in (t_i, t_{i+1})$. Since $h = t_{i+1} - t_i$ and $y'(t_i) = f(t_i, y(t_i))$,

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} h''(\xi_i).$$

Neglecting the remainder term gives Euler's method for $w_i \approx y(t_i)$:

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + hf(t_i, w_i), \qquad i = 0, 1, \ldots, N-1 \end{aligned}$$

The well-posedness implies that

$$f(t_i, w_i) \approx y'(t_i) = f(t_i, y(t_i))$$

## Error Bound

**Theorem**

*Suppose $f$ is continuous and satisfies a Lipschitz condition with constant $L$ on*

$$D = \{(t, y) \mid a \le t \le b, -\infty < y < \infty\}$$

*and that a constant $M$ exists with*

$$|y''(t)| \le M, \quad \text{for all } t \in [a, b].$$

*Let $y(t)$ denote the unique solution to the initial-value problem*

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

*and $w_0, w_1, \ldots, w_n$ as in Euler's method. Then*

$$|y(t_i) - w_i| \le \frac{hM}{2L} \left[ e^{L(t_i - a)} - 1 \right].$$

## Local Truncation Error

**Definition**

The difference method

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h\phi(t_i, w_i) \end{aligned}$$

has *local truncation error*

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\phi(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i),$$

for each $i = 0, 1, \ldots, N-1$.

## Higher-Order Taylor Methods

Consider initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha.$$

Expand $y(t)$ in $n$th Taylor polynomial about $t_i$, evaluated at $t_{i+1}$:

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(t_i) + \cdots \\ &\quad + \frac{h^n}{n!} y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i) \\ &= y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(t_i, y(t_i)) + \cdots \\ &= \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)) \end{aligned}$$

for some $\xi_i \in (t_i, t_{i+1})$. Delete remainder term to obtain the Taylor method of order $n$.

## Higher-Order Taylor Methods

**Taylor Method of Order $n$**

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + hT^{(n)}(t_i, w_i), \qquad i = 0, 1, \ldots, N-1 \end{aligned}$$

where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{(n-1)}}{n!} f^{(n-1)}(t_i, w_i)$$

## Higher-Order Taylor Methods

**Theorem**

*If Taylor's method of order $n$ is used to approximate the solution to*

$$y'(t) = f(t, y(t)), \quad a \le t \le b, \quad y(a) = \alpha,$$

*with step size $h$ and if $y \in C^{n+1}[a, b]$, then the local truncation error is $O(h^n)$.*

## Taylor's Theorem in Two Variables

**Theorem**

*Suppose $f(t,y)$ and partial derivatives up to order $n+1$ continuous on $D = \{(t,y) \mid a \le t \le b, c \le y \le d\}$, let $(t_0, y_0) \in D$. For $(t,y) \in D$, there is $\xi \in [t, t_0]$ and $\mu \in [y, y_0]$ with*

$$
\begin{aligned}
f(t,y) &= P_n(t,y) + R_n(t,y) \\
P_n(t,y) &= f(t_0,y_0) + \left[(t-t_0)\frac{\partial f}{\partial t}(t_0,y_0) + (y-y_0)\frac{\partial f}{\partial y}(t_0,y_0)\right] \\
&\quad + \left[\frac{(t-t_0)^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0,y_0) + (t-t_0)(y-y_0)\frac{\partial^2 f}{\partial t \partial y}(t_0,y_0)\right. \\
&\quad \left. + \frac{(y-y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0,y_0)\right] + \cdots \\
&\quad + \left[\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(t-t_0)^{n-j}(y-y_0)^j \frac{\partial^n f}{\partial t^{n-j}\partial y^j}(t_0,y_0)\right]
\end{aligned}
$$

## Taylor's Theorem in Two Variables

**Theorem**

*(cont'd)*

$$
\begin{aligned}
R_n(t,y) = {} & \frac{1}{(n+1)!}\sum_{j=0}^{n+1}\binom{n+1}{j}(t-t_0)^{n+1-j}(y-y_0)^j \cdot \\
& \cdot \frac{\partial^{n+1} f}{\partial t^{n+1-j}\partial y^j}(\xi,\mu)
\end{aligned}
$$

*$P_n(t,y)$ is the $n$th Taylor polynomial in two variables.*

## Runge-Kutta Methods

- Obtain high-order accuracy of Taylor methods without knowledges of derivatives of $f$
- Determine $a_1, \alpha_1, \beta_1$ such that

$$a_1 f(t+\alpha_1, y+\beta_1) \approx f(t,y) + \frac{h}{2}f'(t,y) = T^{(2)}(t,y).$$

  with $O(h^2)$ error.
- Since

$$f'(t,y) = \frac{df}{dt}(t,y) = \frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y)\cdot y'(t)$$

  and $y'(t) = f(t,y)$, we have

$$T^{(2)}(t,y) = f(t,y) + \frac{h}{2}\frac{\partial f}{\partial t}(t,y) + \frac{h}{2}\frac{\partial f}{\partial y}(t,y)\cdot f(t,y)$$

## Runge-Kutta Methods

- Expand $f(t+\alpha_1, y+\beta_1)$ in 1st degree Taylor polynomial:

$$
\begin{aligned}
a_1 f(t+\alpha_1, y+\beta_1) &= a_1 f(t,y) + a_1\alpha_1\frac{\partial f}{\partial t}(t,y) \\
&\quad + a_1\beta_1\frac{\partial f}{\partial y}(t,y) + a_1\cdot R_1(t+\alpha_1, y+\beta_1)
\end{aligned}
$$

- Matching coefficients gives

$$a_1 = 1 \quad a_1\alpha_1 = \frac{h}{2}, \quad a_1\beta_1 = \frac{h}{2}f(t,y)$$

  with unique solution

$$a_1 = 1, \quad \alpha_1 = \frac{h}{2}, \quad \beta_1 = \frac{h}{2}f(t,y)$$

## Runge-Kutta Methods

- This gives

$$T^{(2)}(t,y) = f\left(t+\frac{h}{2}, y+\frac{h}{2}f(t,y)\right) - R_1\left(t+\frac{h}{2}, y+\frac{h}{2}f(t,y)\right)$$

  with $R_1(\cdot,\cdot) = O(h^2)$

**Midpoint Method**

$$
\begin{aligned}
w_0 &= \alpha, \\
w_{i+1} &= w_i + hf\left(t+\frac{h}{2}, w_i + \frac{h}{2}f(t_i,w_i)\right), \qquad i = 0,1,\ldots,N-1
\end{aligned}
$$

Local truncation error of order two.

## Runge-Kutta Methods

**Runge-Kutta Order Four**

$$
\begin{aligned}
w_0 &= \alpha \\
k_1 &= hf(t_i, w_i) \\
k_2 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right) \\
k_3 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right) \\
k_4 &= hf(t_{i+1}, w_i + k_3) \\
w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}
$$

Local truncation error $O(h^4)$

## Runge-Kutta Order Four

**MATLAB Implementation**

```
function [t,w]=rk4(f,a,b,alpha,N)

h=(b-a)/N;
t=a:h:b;
w=zeros(length(alpha),N+1);
w(:,1)=alpha;
for i=1:N
  k1=h*f(t(i),w(:,i));
  k2=h*f(t(i)+h/2,w(:,i)+k1/2);
  k3=h*f(t(i)+h/2,w(:,i)+k2/2);
  k4=h*f(t(i)+h,w(:,i)+k3);
  w(:,i+1)=w(:,i)+(k1+2*k2+2*k3+k4)/6;
end
```

## Local Truncation Error Estimation

Use $n$th-order and $(n+1)$th-order Taylor or Runge-Kutta methods:

$$y(t_{i+1}) = y(t_i) + h\phi(t_i, y(t_i), h) + O(h^{n+1})$$
$$y(t_{i+1}) = y(t_i) + h\tilde{\phi}(t_i, y(t_i), h) + O(h^{n+2})$$

producing approximations

$$w_0 = \alpha \qquad \tilde{w}_0 = \alpha$$
$$w_{i+1} = w_i + h\phi(t_i, w_i, h) \qquad \tilde{w}_{i+1} = \tilde{w}_i + h\tilde{\phi}(t_i, \tilde{w}_i, h)$$

with local truncation errors $\tau_{i+1}(h) = O(h^n)$ and $\tilde{\tau}_{i+1}(h) = O(h^{n+1})$. Assuming $w_i = y(t_i) = \tilde{w}_i$, we have

$$\tau_{i+1}(h) = \frac{1}{h}(y(t_{i+1}) - w_{i+1}) \qquad \tilde{\tau}_{i+1}(h) = \frac{1}{h}(y(t_{i+1}) - \tilde{w}_{i+1})$$

## Local Truncation Error Estimation

We then have

$$\tau_{i+1}(h) = \tilde{\tau}_{i+1}(h) + \frac{1}{h}(\tilde{w}_{i+1} - w_{i+1}) \approx \frac{1}{h}(\tilde{w}_{i+1} - w_{i+1})$$

since $\tau_{i+1}(h)$ is $O(h^n)$ and $\tilde{\tau}_{i+1}(h)$ is $O(h^{n+1})$. Now find a new step size $qh$ with $\tau_{i+1}(qh)$ bounded by $\varepsilon$:

$$\tau_{i+1}(h) \approx Kh^n \Longrightarrow \tau_{i+1}(qh) \approx K(qh)^n = q^n(Kh^n) \approx q^n \tau_{i+1}(h)$$

Choose $q$ so that

$$\frac{q^n}{h}|\tilde{w}_{i+1} - w_{i+1}| \approx |\tau_{i+1}(qh)| \leq \varepsilon$$

or

$$q \leq \left( \frac{\varepsilon h}{|\tilde{w}_{i+1} - w_{i+1}|} \right)^{1/n}$$

## The Runge-Kutta-Fehlberg Method

- Use a Runge-Kutta method with local truncation error $\tilde{w}_{i+1}$ of order five to estimate the local error $w_{i+1}$ in a Runge-Kutta method of order four
- Use specialized schemes that only require six evaluations of $f$
- Step size control strategy:
  - When $q < 1$, reject the step and repeat calculations with $qh$
  - When $q \geq 1$, accept the step and change step size to $qh$

## Multistep Methods

**Definition**

An *$m$-step multistep method* for solving the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

has a difference equation for approximate $w_{i+1}$ at $t_{i+1}$:

$$\begin{aligned} w_{i+1} &= a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} \\ &+ h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots \\ &+ b_0 f(t_{i+1-m}, w_{i+1-m})], \end{aligned}$$

where $h = (b-a)/N$, and starting values are specified:

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad \ldots, \quad w_{m-1} = \alpha_{m-1}$$

*Explicit* method if $b_m = 0$, *implicit* method if $b_m \neq 0$.

## Multistep Methods

**Fourth-Order Adams-Bashforth Technique**

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3,$$
$$\begin{aligned} w_{i+1} = w_i + \frac{h}{24}[&55 f(t_i, w_i) - 59 f(t_{i-1}, w_{i-1}) \\ &+ 37 f(t_{i-2}, w_{i-2}) - 9 f(t_{i-3}, w_{i-3})] \end{aligned}$$

**Fourth-Order Adams-Moulton Technique**

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2,$$
$$\begin{aligned} w_{i+1} = w_i + \frac{h}{24}[&9 f(t_{i+1}, w_{i+1}) + 19 f(t_i, w_i) \\ &- 5 f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})] \end{aligned}$$

## Derivation of Multistep Methods

Integrate the initial-value problem

$$y' = f(t,y), \quad a \le t \le b, \quad y(a) = \alpha$$

over $[t_i, t_{i+1}]$:

$$y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t))\, dt$$

Replace $f$ by polynomial $P(t)$ interpolating $(t_0, w_0), \dots, (t_i, w_i)$, and approximate $y(t_i) \approx w_i$:

$$y(t_{i+1}) \approx w_i + \int_{t_i}^{t_{i+1}} P(t)\, dt$$

## Derivation of Multistep Methods

Adams-Bashforth explicit $m$-step: Newton backward-difference polynomial through $(t_i, f(t_i, y(t_i))), \dots, (t_{i+1-m}, f(t_{i+1-m}, y(t_{i+1-m})))$.

$$\int_{t_i}^{t_{i+1}} f(t, y(t))\, dt \approx \int_{t_i}^{t_{i+1}} \sum_{k=0}^{m-1} (-1)^k \binom{-s}{k} \nabla^k f(t_i, y(t_i))\, dt$$
$$= \sum_{k=0}^{m-1} \nabla^k f(t_i, y(t_i)) h (-1)^k \int_0^1 \binom{-s}{k}\, ds$$

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $(-1)^k \int_0^1 \binom{-s}{k}\, ds$ | 1 | $\frac{1}{2}$ | $\frac{5}{12}$ | $\frac{3}{8}$ | $\frac{251}{720}$ | $\frac{95}{288}$ |

## Derivation of Multistep Methods

Three-step Adams-Bashforth:

$$y(t_{i+1}) \approx y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}\nabla f(t_i, y(t_i)) + \frac{5}{12}\nabla^2 f(t_i, y(t_i))\right]$$
$$= y(t_i) + \frac{h}{12}[23 f(t_i, y(t_i)) - 16 f(t_{i-1}, y(t_{i-1})) + 5 f(t_{i-2}, y(t_{i-2}))]$$

The method is:

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2,$$
$$w_{i+1} = w_i + \frac{h}{12}[23 f(t_i, w_i) - 16 f(t_{i-1}, w_{i-1}) + 5 f(t_{i-2}, w_{i-2})]$$

## Local Truncation Error

**Definition**

If $y(t)$ solves

$$y' = f(t,y), \quad a \le t \le b, \quad y(a) = \alpha,$$

and

$$w_{i+1} = a_{m-1} w_i + \cdots + a_0 w_{i+1-m} + h[b_m f(t_{i+1}, w_{i+1}) + \cdots + b_0 f(t_{i+1-m}, w_{i+1-m})],$$

the *local truncation error* is

$$\tau_{i+1}(h) = \frac{y(t_{i+1}) - a_{m-1} y(t_i) - \cdots - a_0 y(t_{i+1-m})}{h} - [b_m f(t_{i+1}, y(t_{i+1})) + \cdots + b_0 f(t_{i+1-m}, y(t_{i+1-m}))].$$

## Predictor-Corrector Methods

- Calculate approximation $w_4^{(0)}$ using the explicit Adams-Bashforth method (*predictor*):

$$w_4^{(0)} = w_3 + \frac{h}{24}[55 f(t_3, w_3) - 59 f(t_2, w_2) + 37 f(t_1, w_1) - 9 f(t_0, w_0)]$$

- Insert in the right side of the implicit Adams-Moulton method (*corrector*):

$$w_4^{(1)} = w_3 + \frac{h}{24}[9 f(t_4, w_4^{(0)}) + 19 f(t_3, w_3) - 5 f(t_2, w_2) + f(t_1, w_1)]$$

- Iterations $w_4^{(k)}$ can be computed, but one step is usually sufficient

## Error Control

- Use the predictor-corrector method for error estimation
- Adams-Bashforth and Adams-Moulton truncation errors:

$$\frac{y(t_{i+1}) - w_{i+1}^{(0)}}{h} = \frac{251}{720} y^{(5)}(\hat{\mu}_i) h^4$$
$$\frac{y(t_{i+1}) - w_{i+1}}{h} = -\frac{19}{720} y^{(5)}(\tilde{\mu}_i) h^4$$

- Assuming $y^{(5)}(\hat{\mu}_i) \approx y^{(5)}(\tilde{\mu}_i)$, we have

$$\frac{w_{i+1} - w_{i+1}^{(0)}}{h} \approx \frac{3}{8} h^4 y^{(5)}(\tilde{\mu}_i)$$

or

$$y^{(5)}(\tilde{\mu}_i) \approx \frac{8}{3h^5}(w_{i+1} - w_{i+1}^{(0)})$$

## Error Control

- The local truncation error of the Adams-Moulton method is then

$$|\tau_{i+1}(h)| \approx \frac{19|w_{i+1} - w_{i+1}^{(0)}|}{270h}$$

- Compute a new step size $qh$ such that the error is $\varepsilon$:

$$q < \left( \frac{270}{19} \frac{h\varepsilon}{|w_{i+1} - w_{i+1}^{(0)}|} \right)^{1/4}$$

- Step size changes for multistep methods are expensive, so keep same step size if

$$\frac{\varepsilon}{10} < |\tau_{i+1}(h)| < \varepsilon$$

## Extrapolation Methods

Consider the approximation of $y(a+h)$ in the initial-value problem

$$y'(t) = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

First set $h_0 = h/2$ and use Euler, Midpoint, and endpoint correction:

$$\begin{aligned} w_1 &= w_0 + h_0 f(a, w_0) \\ w_2 &= w_0 + 2h_0 f(a + h_0, w_1) \\ y_{1,1} &= \frac{1}{2}[w_2 + w_1 + h_0 f(a + 2h_0, w_2)] \end{aligned}$$

Then set $h_1 = h/4$ and repeat to obtain $y_{2,1}$

## Extrapolation Methods

The two approximations to $y(a+h)$ are

$$\begin{aligned} y(a+h) &= y_{1,1} + \delta_1 \left(\frac{h}{2}\right)^2 + \delta_2 \left(\frac{h}{2}\right)^4 + \cdots \\ y(a+h) &= y_{2,1} + \delta_1 \left(\frac{h}{4}\right)^2 + \delta_2 \left(\frac{h}{4}\right)^4 + \cdots \end{aligned}$$

Eliminate $O(h^2)$ portion by subtracting first from 4 times second:

$$y(a+h) = y_{2,1} + \frac{1}{3}(y_{2,1} - y_{1,1}) - \delta_2 \frac{h^4}{64} + \cdots$$

This gives an error of order $O(h^4)$ in the approximation

$$y_{2,2} = y_{2,1} + \frac{1}{3}(y_{2,1} - y_{1,1})$$

## Extrapolation Methods

Apply the technique with step sizes $h_i = h/q_i$, where

$$q_0 = 2, q_1 = 4, q_2 = 6, q_3 = 8, q_4 = 12, q_5 = 16, q_6 = 24, q_7 = 32$$

Build table

$$\begin{array}{lll} y_{1,1} = w(t, h_0) & & \\ y_{2,1} = w(t, h_1) & y_{2,2} = y_{2,1} + \frac{h_1^2}{h_0^2 - h_1^2}(y_{2,1} - y_{1,1}) & \\ y_{3,1} = w(t, h_2) & y_{3,2} = y_{3,1} + \frac{h_2^2}{h_1^2 - h_2^2}(y_{3,1} - y_{2,1}) & y_{3,3} = y_{3,2} + \frac{h_2^2}{h_0^2 - h_2^2}(y_{3,2} - y_{2,2}) \end{array}$$

## High-Order Systems of Initial-Value Problems

An *mth-order system* of first-order initial-value problems has the form

$$\begin{aligned} \frac{du_1}{dt}(t) &= f_1(t, u_1, u_2, \ldots, u_m), \\ \frac{du_2}{dt}(t) &= f_2(t, u_1, u_2, \ldots, u_m), \\ &\vdots \\ \frac{du_m}{dt}(t) &= f_m(t, u_1, u_2, \ldots, u_m), \end{aligned}$$

for $a \le t \le b$, with the initial conditions

$$u_1(a) = \alpha_1, u_2(a) = \alpha_2, \ldots, u_m(a) = \alpha_m.$$

## Existence and Uniqueness

**Definition**

The function $f(t, y_1, \ldots, y_m)$, defined on the set

$$D = \{(t, u_1, \ldots, u_m) \mid a \le t \le b, -\infty < u_i < \infty, i = 1, 2, \ldots, m\}$$

is said to satisfy a *Lipschitz condition* on $D$ in the variables $u_1, u_2, \ldots, u_m$ if a constant $L > 0$ exists with

$$|f(t, u_1, \ldots, u_m) - f(t, z_1, \ldots, z_m)| \le L \sum_{j=1}^{m} |u_j - z_j|,$$

for all $(t, u_1, \ldots, u_m)$ and $(t, z_1, \ldots, z_m)$ in $D$.

## Existence and Uniqueness

**Theorem**

*Suppose*

$$D = \{(t, u_1, \ldots, u_m) \mid a \le t \le b, -\infty < u_i < \infty, i = 1, 2, \ldots, m\}$$

*and let $f_i(t, u_1, \ldots, u_m)$, for each $i = 1, 2, \ldots, m$, be continuous on $D$ and satisfy a Lipschitz condition there. The system of first-order differential equations*

$$\frac{du_k}{dt}(t) = f_k(t, u_1, \ldots, u_m), \quad u_k(a) = \alpha_k, \quad k = 1, \ldots, m$$

*has a unique solution $u_1(t), \ldots, u_m(t)$ for $a \le t \le b$.*

## Numerical Methods

Numerical methods for systems of first-order differential equations are vector-valued generalizations of methods for single equations.

**Fourth order Runge-Kutta for systems**

$$\begin{aligned}
\boldsymbol{w}_0 &= \boldsymbol{\alpha} \\
\boldsymbol{k}_1 &= h\boldsymbol{f}(t_i, \boldsymbol{w}_i) \\
\boldsymbol{k}_2 &= h\boldsymbol{f}(t_i + \frac{h}{2}, \boldsymbol{w}_i + \frac{1}{2}\boldsymbol{k}_1) \\
\boldsymbol{k}_3 &= h\boldsymbol{f}(t_i + \frac{h}{2}, \boldsymbol{w}_i + \frac{1}{2}\boldsymbol{k}_2) \\
\boldsymbol{k}_4 &= h\boldsymbol{f}(t_{i+1}, \boldsymbol{w}_i + \boldsymbol{k}_3) \\
\boldsymbol{w}_{i+1} &= \boldsymbol{w}_i + \frac{1}{6}(\boldsymbol{k}_1 + 2\boldsymbol{k}_2 + 2\boldsymbol{k}_3 + \boldsymbol{k}_4)
\end{aligned}$$

where $\boldsymbol{w}_i = (w_{i,1}, \ldots, w_{i,m})$ is the vector of unknowns.

## Consistency and Convergence

**Definition**

A one-step difference-equation with local truncation error $\tau_i(h)$ is said to be *consistent* if

$$\lim_{h\to 0} \max_{1\le i\le N} |\tau_i(h)| = 0$$

**Definition**

A one-step difference equation is said to be *convergent* if

$$\lim_{h\to 0} \max_{1\le i\le N} |w_i - y(t_i)| = 0,$$

where $y_i = y(t_i)$ is the exact solution and $w_i$ the approximation.

## Convergence of One-Step Methods

**Theorem**

*Suppose the initial-value problem $y' = f(t, y)$, $a \le t \le b$, $y(a) = \alpha$ is approximated by a one-step difference method in the form $w_0 = \alpha$, $w_{i+1} = w_i + h\phi(t_i, w_i, h)$. Suppose also that $h_0 > 0$ exists and $\phi(t, w, h)$ is continuous with a Lipschitz condition in $w$ with constant $L$ on $D$, then*

$$D = \{(t, w, h) \mid a \le t \le b, -\infty < w < \infty, 0 \le h \le h_0\}.$$

1. *The method is stable;*
2. *The method is convergent if and only if it is consistent:*

$$\phi(t, y, 0) = f(t, y)$$

3. *If $\tau$ exists s.t. $|\tau_i(h)| \le \tau(h)$ when $0 \le h \le h_0$, then*

$$|y(t_i) - w_i| \le \frac{\tau(h)}{L} e^{L(t_i - a)}.$$

## Root Condition

**Definition**

Let $\lambda_1, \ldots, \lambda_m$ denote the roots of the characteristic equation

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - \cdots - a_1\lambda - a_0 = 0$$

associated with the multistep method

$$\begin{aligned}
w_{i+1} = & a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} \\
& + hF(t_i, h, w_{i+1}, w_i, \ldots, w_{i+1-m}).
\end{aligned}$$

If $|\lambda_i| \le 1$ and all roots with absolute value 1 are simple, the method is said to satisfy the *root condition*.

## Stability

**Definition**

1. Methods that satisfy the root condition and have $\lambda = 1$ as the only root of magnitude one are called *strongly stable*.
2. Methods that satisfy the root condition and have more than one distinct root with magnitude one are called *weakly stable*.
3. Methods that do not satisfy the root condition are *unstable*.

**Theorem**

*A multistep method*

$$\begin{aligned}
w_{i+1} = & a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} \\
& + hF(t_i, h, w_{i+1}, w_i, \ldots, w_{i+1-m})
\end{aligned}$$

*is stable if and and only if it satisfies the root condition. If it is also consistent, then it is stable if and only if it is convergent.*

## Stiff Equations

- A *stiff differential equation* is numerically unstable unless the step size is extremely small
- Large derivatives give error terms that are dominating the solution
- Example: The initial-value problem

$$y' = -30y, \quad 0 \le t \le 1.5, \quad y(0) = \frac{1}{3}$$

has exact solution $y = \frac{1}{3}e^{-30t}$. But RK4 is unstable with step size $h = 0.1$ .

## Euler's Method for Test Equation

- Consider the simple test equation

$$y' = \lambda y, \quad y(0) = \alpha, \quad \text{where } \lambda < 0$$

with solution $y(t) = \alpha e^{\lambda t}$.

- Euler's method gives $w_0 = \alpha$ and

$$w_{j+1} = w_j + h(\lambda w_j) = (1 + h\lambda)w_j = (1 + h\lambda)^{j+1}\alpha.$$

- The absolute error is

$$\begin{aligned} |y(t_j) - w_j| &= \left| e^{jh\lambda} - (1 + h\lambda)^j \right| |\alpha| \\ &= \left| (e^{h\lambda})^j - (1 + h\lambda)^j \right| |\alpha| \end{aligned}$$

- Stability requires $|1 + h\lambda| < 1$, or $h < 2/|\lambda|$.

## Multistep Methods

Apply a multistep method to the test equation:

$$\begin{aligned} w_{j+1} = &a_{m-1}w_j + \cdots + a_0 w_{j+1-m} \\ &+ h\lambda(b_m w_{j+1} + b_{m-1}w_j + \cdots + b_0 w_{j+1-m}) \end{aligned}$$

or

$$(1 - h\lambda b_m)w_{j+1} - (a_{m-1} + h\lambda b_{m-1})w_j - \cdots - (a_0 + h\lambda b_0)w_{j+1-m} = 0$$

Let $\beta_1, \ldots, \beta_m$ be the zeros of the *characteristic polynomial*

$$Q(z, h\lambda) = (1 - h\lambda b_m)z^m - (a_{m-1} + h\lambda b_{m-1})z^{m-1} - \cdots - (a_0 + h\lambda b_0)$$

Then $c_1, \ldots, c_m$ exist with

$$w_j = \sum_{k=1}^{m} c_k(\beta_k)^j$$

and $|\beta_k| < 1$ is required for stability.

## Region of Stability

**Definition**

The *region $R$ of absolute stability* for a one-step method is $R = \{h\lambda \in \mathcal{C} \mid |Q(h\lambda)| < 1\}$, and for a multistep method, it is $R = \{h\lambda \in \mathcal{C} \mid |\beta_k| < 1, \text{ for all zeros } \beta_k \text{ of } Q(z, h\lambda)\}$.

A numerical method is said to be *A-stable* is its region $R$ of absolute stability contains the entire left half-plane.