

Chapter 5 – Initial-Value Problems for Ordinary Differential Equations

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Definition

A function $f(t, y)$ is said to satisfy a *Lipschitz condition* in the variable y on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists with

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|,$$

whenever $(t, y_1), (t, y_2) \in D$. The constant L is called a *Lipschitz constant* for f .

Definition

A set $D \subset \mathbb{R}^2$ is said to be *convex* if whenever (t_1, y_1) and (t_2, y_2) belong to D and λ is in $[0, 1]$, the point $((1 - \lambda)t_1 + \lambda t_2, (1 - \lambda)y_1 + \lambda y_2)$ also belongs to D .

Existence and Uniqueness

Theorem

Suppose $f(t, y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L > 0$ exists with

$$\left| \frac{\partial f}{\partial y}(t, y) \right| \leq L, \quad \text{for all } (t, y) \in D,$$

then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L .

Theorem

Suppose that $D = \{(t, y) \mid a \leq t \leq b, -\infty < y < \infty\}$ and that $f(t, y)$ is continuous on D . If f satisfies a Lipschitz condition on D in the variable y , then the initial-value problem

$$y'(t) = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

has a unique solution $y(t)$ for $a \leq t \leq b$.

Definition

The initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

is said to be a *well-posed problem* if:

- A unique solution, $y(t)$, to the problem exists, and
- There exist constants $\varepsilon_0 > 0$ and $k > 0$ such that for any ε , with $\varepsilon_0 > \varepsilon > 0$, whenever $\delta(t)$ is continuous with $|\delta(t)| < \varepsilon$ for all t in $[a, b]$, and when $|\delta_0| < \varepsilon$, the initial-value problem

$$\frac{dz}{dt} = f(t, z) + \delta(t), \quad a \leq t \leq b, \quad z(a) = \alpha + \delta_0,$$

has a unique solution $z(t)$ that satisfies

$$|z(t) - y(t)| < k\varepsilon \quad \text{for all } t \text{ in } [a, b].$$

Theorem

Suppose $D = \{(t, y) \mid a \leq t \leq b \text{ and } -\infty < y < \infty\}$. If f is continuous and satisfies a Lipschitz condition in the variable y on the set D , then the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

is well-posed.

Suppose a well-posed initial-value problem is given:

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

Distribute mesh points equally throughout $[a, b]$:

$$t_i = a + ih, \quad \text{for each } i = 0, 1, 2, \dots, N.$$

The *step size* $h = (b - a)/N = t_{i+1} - t_i$.

Euler's Method

Use Taylor's Theorem for $y(t)$:

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2}y''(\xi_i)$$

for $\xi_i \in (t_i, t_{i+1})$. Since $h = t_{i+1} - t_i$ and $y'(t_i) = f(t_i, y(t_i))$,

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}y''(\xi_i).$$

Neglecting the remainder term gives Euler's method for $w_i \approx y(t_i)$:

$$w_0 = \alpha$$

$$w_{i+1} = w_i + hf(t_i, w_i), \quad i = 0, 1, \dots, N - 1$$

The well-posedness implies that

$$f(t_i, w_i) \approx y'(t_i) = f(t_i, y(t_i))$$

Theorem

Suppose f is continuous and satisfies a Lipschitz condition with constant L on

$$D = \{(t, y) \mid a \leq t \leq b, -\infty < y < \infty\}$$

and that a constant M exists with

$$|y''(t)| \leq M, \quad \text{for all } t \in [a, b].$$

Let $y(t)$ denote the unique solution to the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

and w_0, w_1, \dots, w_n as in Euler's method. Then

$$|y(t_i) - w_i| \leq \frac{hM}{2L} \left[e^{L(t_i-a)} - 1 \right].$$

Definition

The difference method

$$\begin{aligned}w_0 &= \alpha \\w_{i+1} &= w_i + h\phi(t_i, w_i)\end{aligned}$$

has *local truncation error*

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\phi(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i),$$

for each $i = 0, 1, \dots, N - 1$.

Higher-Order Taylor Methods

Consider initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha.$$

Expand $y(t)$ in n th Taylor polynomial about t_i , evaluated at t_{i+1} :

$$\begin{aligned}y(t_{i+1}) &= y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots \\ &+ \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i) \\ &= y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}f'(t_i, y(t_i)) + \cdots \\ &= \frac{h^n}{n!}f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!}f^{(n)}(\xi_i, y(\xi_i))\end{aligned}$$

for some $\xi_i \in (t_i, t_{i+1})$. Delete remainder term to obtain the Taylor method of order n .

Taylor Method of Order n

$$w_0 = \alpha$$

$$w_{i+1} = w_i + hT^{(n)}(t_i, w_i), \quad i = 0, 1, \dots, N - 1$$

where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) + \dots + \frac{h^{(n-1)}}{n!}f^{(n-1)}(t_i, w_i)$$

Theorem

If Taylor's method of order n is used to approximate the solution to

$$y'(t) = f(t, y(t)), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

with step size h and if $y \in C^{n+1}[a, b]$, then the local truncation error is $O(h^n)$.

Taylor's Theorem in Two Variables

Theorem

Suppose $f(t, y)$ and partial derivatives up to order $n + 1$ continuous on $D = \{(t, y) \mid a \leq t \leq b, c \leq y \leq d\}$, let $(t_0, y_0) \in D$. For $(t, y) \in D$, there is $\xi \in [t, t_0]$ and $\mu \in [y, y_0]$ with

$$f(t, y) = P_n(t, y) + R_n(t, y)$$

$$\begin{aligned} P_n(t, y) = & f(t_0, y_0) + \left[(t - t_0) \frac{\partial f}{\partial t}(t_0, y_0) + (y - y_0) \frac{\partial f}{\partial y}(t_0, y_0) \right] \\ & + \left[\frac{(t - t_0)^2}{2} \frac{\partial^2 f}{\partial t^2}(t_0, y_0) + (t - t_0)(y - y_0) \frac{\partial^2 f}{\partial t \partial y}(t_0, y_0) \right. \\ & \left. + \frac{(y - y_0)^2}{2} \frac{\partial^2 f}{\partial y^2}(t_0, y_0) \right] + \dots \\ & + \left[\frac{1}{n!} \sum_{j=0}^n \binom{n}{j} (t - t_0)^{n-j} (y - y_0)^j \frac{\partial^n f}{\partial t^{n-j} \partial y^j}(t_0, y_0) \right] \end{aligned}$$

Theorem

(cont'd)

$$R_n(t, y) = \frac{1}{(n+1)!} \sum_{j=0}^{n+1} \binom{n+1}{j} (t-t_0)^{n+1-j} (y-y_0)^j \cdot \frac{\partial^{n+1} f}{\partial t^{n+1-j} \partial y^j}(\xi, \mu)$$

$P_n(t, y)$ is the n th Taylor polynomial in two variables.

Runge-Kutta Methods

- Obtain high-order accuracy of Taylor methods without knowledges of derivatives of f
- Determine a_1, α_1, β_1 such that

$$a_1 f(t + \alpha_1, y + \beta_1) \approx f(t, y) + \frac{h}{2} f'(t, y) = T^{(2)}(t, y).$$

with $O(h^2)$ error.

- Since

$$f'(t, y) = \frac{df}{dt}(t, y) = \frac{\partial f}{\partial t}(t, y) + \frac{\partial f}{\partial y}(t, y) \cdot y'(t)$$

and $y'(t) = f(t, y)$, we have

$$T^{(2)}(t, y) = f(t, y) + \frac{h}{2} \frac{\partial f}{\partial t}(t, y) + \frac{h}{2} \frac{\partial f}{\partial y}(t, y) \cdot f(t, y)$$

- Expand $f(t + \alpha_1, y + \beta_1)$ in 1st degree Taylor polynomial:

$$\begin{aligned} a_1 f(t + \alpha_1, y + \beta_1) &= a_1 f(t, y) + a_1 \alpha_1 \frac{\partial f}{\partial t}(t, y) \\ &\quad + a_1 \beta_1 \frac{\partial f}{\partial y}(t, y) + a_1 \cdot R_1(t + \alpha_1, y + \beta_1) \end{aligned}$$

- Matching coefficients gives

$$a_1 = 1 \quad a_1 \alpha_1 = \frac{h}{2}, \quad a_1 \beta_1 = \frac{h}{2} f(t, y)$$

with unique solution

$$a_1 = 1, \quad \alpha_1 = \frac{h}{2}, \quad \beta_1 = \frac{h}{2} f(t, y)$$

- This gives

$$T^{(2)}(t, y) = f\left(t + \frac{h}{2}, y + \frac{h}{2}f(t, y)\right) - R_1\left(t + \frac{h}{2}, y + \frac{h}{2}f(t, y)\right)$$

with $R_1(\cdot, \cdot) = O(h^2)$

Midpoint Method

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + hf\left(t + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)\right), \quad i = 0, 1, \dots, N - 1$$

Local truncation error of order two.

Runge-Kutta Order Four

$$w_0 = \alpha$$

$$k_1 = hf(t_i, w_i)$$

$$k_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right)$$

$$k_4 = hf(t_{i+1}, w_i + k_3)$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Local truncation error $O(h^4)$

Runge-Kutta Order Four

MATLAB Implementation

```
function [t,w]=rk4(f,a,b,alpha,N)

h=(b-a)/N;
t=a:h:b;
w=zeros(length(alpha),N+1);
w(:,1)=alpha;
for i=1:N
    k1=h*f(t(i),w(:,i));
    k2=h*f(t(i)+h/2,w(:,i)+k1/2);
    k3=h*f(t(i)+h/2,w(:,i)+k2/2);
    k4=h*f(t(i)+h,w(:,i)+k3);
    w(:,i+1)=w(:,i)+(k1+2*k2+2*k3+k4)/6;
end
```

Local Truncation Error Estimation

Use n th-order and $(n + 1)$ th-order Taylor or Runge-Kutta methods:

$$y(t_{i+1}) = y(t_i) + h\phi(t_i, y(t_i), h) + O(h^{n+1})$$

$$y(t_{i+1}) = y(t_i) + h\tilde{\phi}(t_i, y(t_i), h) + O(h^{n+2})$$

producing approximations

$$w_0 = \alpha$$

$$\tilde{w}_0 = \alpha$$

$$w_{i+1} = w_i + h\phi(t_i, w_i, h)$$

$$\tilde{w}_{i+1} = \tilde{w}_i + h\tilde{\phi}(t_i, \tilde{w}_i, h)$$

with local truncation errors $\tau_{i+1}(h) = O(h^n)$ and $\tilde{\tau}_{i+1}(h) = O(h^{n+1})$. Assuming $w_i = y(t_i) = \tilde{w}_i$, we have

$$\tau_{i+1}(h) = \frac{1}{h}(y(t_{i+1}) - w_{i+1}) \quad \tilde{\tau}_{i+1}(h) = \frac{1}{h}(y(t_{i+1}) - \tilde{w}_{i+1})$$

Local Truncation Error Estimation

We then have

$$\tau_{i+1}(h) = \tilde{\tau}_{i+1}(h) + \frac{1}{h}(\tilde{w}_{i+1} - w_{i+1}) \approx \frac{1}{h}(\tilde{w}_{i+1} - w_{i+1})$$

since $\tau_{i+1}(h)$ is $O(h^n)$ and $\tilde{\tau}_{i+1}(h)$ is $O(h^{n+1})$. Now find a new step size qh with $\tau_{i+1}(qh)$ bounded by ε :

$$\tau_{i+1}(h) \approx Kh^n \implies \tau_{i+1}(qh) \approx K(qh)^n = q^n(Kh^n) \approx q^n\tau_{i+1}(h)$$

Choose q so that

$$\frac{q^n}{h}|\tilde{w}_{i+1} - w_{i+1}| \approx |\tau_{i+1}(qh)| \leq \varepsilon$$

or

$$q \leq \left(\frac{\varepsilon h}{|\tilde{w}_{i+1} - w_{i+1}|} \right)^{1/n}$$

The Runge-Kutta-Fehlberg Method

- Use a Runge-Kutta method with local truncation error \tilde{w}_{i+1} of order five to estimate the local error w_{i+1} in a Runge-Kutta method of order four
- Use specialized schemes that only require six evaluations of f
- Step size control strategy:
 - When $q < 1$, reject the step and repeat calculations with qh
 - When $q \geq 1$, accept the step and change step size to qh

Definition

An m -step multistep method for solving the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

has a difference equation for approximate w_{i+1} at t_{i+1} :

$$\begin{aligned} w_{i+1} = & a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0w_{i+1-m} \\ & + h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1}f(t_i, w_i) + \cdots \\ & + b_0 f(t_{i+1-m}, w_{i+1-m})], \end{aligned}$$

where $h = (b - a)/N$, and starting values are specified:

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad \dots, \quad w_{m-1} = \alpha_{m-1}$$

Explicit method if $b_m = 0$, *implicit* method if $b_m \neq 0$.

Fourth-Order Adams-Bashforth Technique

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3,$$
$$w_{i+1} = w_i + \frac{h}{24} [55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) \\ + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})]$$

Fourth-Order Adams-Moulton Technique

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2,$$
$$w_{i+1} = w_i + \frac{h}{24} [9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) \\ - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})]$$

Derivation of Multistep Methods

Integrate the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

over $[t_i, t_{i+1}]$:

$$y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t)) dt$$

Replace f by polynomial $P(t)$ interpolating $(t_0, w_0), \dots, (t_i, w_i)$, and approximate $y(t_i) \approx w_i$:

$$y(t_{i+1}) \approx w_i + \int_{t_i}^{t_{i+1}} P(t) dt$$

Derivation of Multistep Methods

Adams-Bashforth explicit m -step: Newton backward-difference polynomial through

$(t_i, f(t_i, y(t_i))), \dots, (t_{i+1-m}, f(t_{i+1-m}, y(t_{i+1-m})))$.

$$\begin{aligned} \int_{t_i}^{t_{i+1}} f(t, y(t)) dt &\approx \int_{t_i}^{t_{i+1}} \sum_{k=0}^{m-1} (-1)^k \binom{-s}{k} \nabla^k f(t_i, y(t_i)) dt \\ &= \sum_{k=0}^{m-1} \nabla^k f(t_i, y(t_i)) h (-1)^k \int_0^1 \binom{-s}{k} ds \end{aligned}$$

k	0	1	2	3	4	5
$(-1)^k \int_0^1 \binom{-s}{k} ds$	1	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{3}{8}$	$\frac{251}{720}$	$\frac{95}{288}$

Derivation of Multistep Methods

Three-step Adams-Bashforth:

$$\begin{aligned}y(t_{i+1}) &\approx y(t_i) + h \left[f(t_i, y(t_i)) + \frac{1}{2} \nabla f(t_i, y(t_i)) + \frac{5}{12} \nabla^2 f(t_i, y(t_i)) \right] \\ &= y(t_i) + \frac{h}{12} [23f(t_i, y(t_i)) - 16f(t_{i-1}, y(t_{i-1})) + 5f(t_{i-2}, y(t_{i-2}))]\end{aligned}$$

The method is:

$$\begin{aligned}w_0 &= \alpha, & w_1 &= \alpha_1, & w_2 &= \alpha_2, \\ w_{i+1} &= w_i + \frac{h}{12} [23f(t_i, w_i) - 16f(t_{i-1}, w_{i-1}) + 5f(t_{i-2}, w_{i-2})]\end{aligned}$$

Local Truncation Error

Definition

If $y(t)$ solves

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

and

$$w_{i+1} = a_{m-1}w_i + \cdots + a_0w_{i+1-m} \\ + h[b_m f(t_{i+1}, w_{i+1}) + \cdots + b_0 f(t_{i+1-m}, w_{i+1-m})],$$

the *local truncation error* is

$$\tau_{i+1}(h) = \frac{y(t_{i+1}) - a_{m-1}y(t_i) - \cdots - a_0y(t_{i+1-m})}{h} \\ - [b_m f(t_{i+1}, y(t_{i+1})) + \cdots + b_0 f(t_{i+1-m}, y(t_{i+1-m}))].$$

Predictor-Corrector Methods

- Calculate approximation $w_4^{(0)}$ using the explicit Adams-Bashforth method (*predictor*):

$$w_4^{(0)} = w_3 + \frac{h}{24} [55f(t_3, w_3) - 59f(t_2, w_2) + 37f(t_1, w_1) - 9f(t_0, w_0)]$$

- Insert in the right side of the implicit Adams-Moulton method (*corrector*):

$$w_4^{(1)} = w_3 + \frac{h}{24} [9f(t_4, w_4^{(0)}) + 19f(t_3, w_3) - 5f(t_2, w_2) + f(t_1, w_1)]$$

- Iterations $w_4^{(k)}$ can be computed, but one step is usually sufficient

- Use the predictor-corrector method for error estimation
- Adams-Bashforth and Adams-Moulton truncation errors:

$$\frac{y(t_{i+1}) - w_{i+1}^{(0)}}{h} = \frac{251}{720}y^{(5)}(\hat{\mu}_i)h^4$$
$$\frac{y(t_{i+1}) - w_{i+1}}{h} = -\frac{19}{720}y^{(5)}(\tilde{\mu}_i)h^4$$

- Assuming $y^{(5)}(\hat{\mu}_i) \approx y^{(5)}(\tilde{\mu}_i)$, we have

$$\frac{w_{i+1} - w_{i+1}^{(0)}}{h} \approx \frac{3}{8}h^4y^{(5)}(\tilde{\mu}_i)$$

or

$$y^{(5)}(\tilde{\mu}_i) \approx \frac{8}{3h^5}(w_{i+1} - w_{i+1}^{(0)})$$

- The local truncation error of the Adams-Moulton method is then

$$|\tau_{i+1}(h)| \approx \frac{19|w_{i+1} - w_{i+1}^{(0)}|}{270h}$$

- Compute a new step size qh such that the error is ε :

$$q < \left(\frac{270}{19} \frac{h\varepsilon}{|w_{i+1} - w_{i+1}^{(0)}|} \right)^{1/4}$$

- Step size changes for multistep methods are expensive, so keep same step size if

$$\frac{\varepsilon}{10} < |\tau_{i+1}(h)| < \varepsilon$$

Extrapolation Methods

Consider the approximation of $y(a+h)$ in the initial-value problem

$$y'(t) = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

First set $h_0 = h/2$ and use Euler, Midpoint, and endpoint correction:

$$w_1 = w_0 + h_0 f(a, w_0)$$

$$w_2 = w_0 + 2h_0 f(a + h_0, w_1)$$

$$y_{1,1} = \frac{1}{2}[w_2 + w_1 + h_0 f(a + 2h_0, w_2)]$$

Then set $h_1 = h/4$ and repeat to obtain $y_{2,1}$

Extrapolation Methods

The two approximations to $y(a + h)$ are

$$y(a + h) = y_{1,1} + \delta_1 \left(\frac{h}{2}\right)^2 + \delta_2 \left(\frac{h}{2}\right)^4 + \dots$$

$$y(a + h) = y_{2,1} + \delta_1 \left(\frac{h}{4}\right)^2 + \delta_2 \left(\frac{h}{4}\right)^4 + \dots$$

Eliminate $O(h^2)$ portion by subtracting first from 4 times second:

$$y(a + h) = y_{2,1} + \frac{1}{3}(y_{2,1} - y_{1,1}) - \delta_2 \frac{h^4}{64} + \dots$$

This gives an error of order $O(h^4)$ in the approximation

$$y_{2,2} = y_{2,1} + \frac{1}{3}(y_{2,1} - y_{1,1})$$

Extrapolation Methods

Apply the technique with step sizes $h_i = h/q_i$, where

$$q_0 = 2, q_1 = 4, q_2 = 6, q_3 = 8, q_4 = 12, q_5 = 16, q_6 = 24, q_7 = 32$$

Build table

$$y_{1,1} = w(t, h_0)$$

$$y_{2,1} = w(t, h_1) \quad y_{2,2} = y_{2,1} + \frac{h_1^2}{h_0^2 - h_1^2}(y_{2,1} - y_{1,1})$$

$$y_{3,1} = w(t, h_2) \quad y_{3,2} = y_{3,1} + \frac{h_2^2}{h_1^2 - h_2^2}(y_{3,1} - y_{2,1}) \quad y_{3,3} = y_{3,2} + \frac{h_2^2}{h_0^2 - h_2^2}(y_{3,2} - y_{2,2})$$

High-Order Systems of Initial-Value Problems

An m th-order system of first-order initial-value problems has the form

$$\begin{aligned}\frac{du_1}{dt}(t) &= f_1(t, u_1, u_2, \dots, u_m), \\ \frac{du_2}{dt}(t) &= f_2(t, u_1, u_2, \dots, u_m), \\ &\vdots \\ \frac{du_m}{dt}(t) &= f_m(t, u_1, u_2, \dots, u_m),\end{aligned}$$

for $a \leq t \leq b$, with the initial conditions

$$u_1(a) = \alpha_1, u_2(a) = \alpha_2, \dots, u_m(a) = \alpha_m.$$

Definition

The function $f(t, y_1, \dots, y_m)$, defined on the set

$$D = \{(t, u_1, \dots, u_m) \mid a \leq t \leq b, -\infty < u_i < \infty, i = 1, 2, \dots, m\}$$

is said to satisfy a *Lipschitz condition* on D in the variables u_1, u_2, \dots, u_m if a constant $L > 0$ exists with

$$|f(t, u_1, \dots, u_m) - f(t, z_1, \dots, z_m)| \leq L \sum_{j=1}^m |u_j - z_j|,$$

for all (t, u_1, \dots, u_m) and (t, z_1, \dots, z_m) in D .

Theorem

Suppose

$$D = \{(t, u_1, \dots, u_m) \mid a \leq t \leq b, -\infty < u_i < \infty, i = 1, 2, \dots, m\}$$

and let $f_i(t, u_1, \dots, u_m)$, for each $i = 1, 2, \dots, m$, be continuous on D and satisfy a Lipschitz condition there. The system of first-order differential equations

$$\frac{du_k}{dt}(t) = f_k(t, u_1, \dots, u_m), \quad u_k(a) = \alpha_k, \quad k = 1, \dots, m$$

has a unique solution $u_1(t), \dots, u_m(t)$ for $a \leq t \leq b$.

Numerical methods for systems of first-order differential equations are vector-valued generalizations of methods for single equations.

Fourth order Runge-Kutta for systems

$$\mathbf{w}_0 = \boldsymbol{\alpha}$$

$$\mathbf{k}_1 = h\mathbf{f}(t_i, \mathbf{w}_i)$$

$$\mathbf{k}_2 = h\mathbf{f}\left(t_i + \frac{h}{2}, \mathbf{w}_i + \frac{1}{2}\mathbf{k}_1\right)$$

$$\mathbf{k}_3 = h\mathbf{f}\left(t_i + \frac{h}{2}, \mathbf{w}_i + \frac{1}{2}\mathbf{k}_2\right)$$

$$\mathbf{k}_4 = h\mathbf{f}(t_{i+1}, \mathbf{w}_i + \mathbf{k}_3)$$

$$\mathbf{w}_{i+1} = \mathbf{w}_i + \frac{1}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4)$$

where $\mathbf{w}_i = (w_{i,1}, \dots, w_{i,m})$ is the vector of unknowns.

Consistency and Convergence

Definition

A one-step difference-equation with local truncation error $\tau_i(h)$ is said to be *consistent* if

$$\lim_{h \rightarrow 0} \max_{1 \leq i \leq N} |\tau_i(h)| = 0$$

Definition

A one-step difference equation is said to be *convergent* if

$$\lim_{h \rightarrow 0} \max_{1 \leq i \leq N} |w_i - y(t_i)| = 0,$$

where $y_i = y(t_i)$ is the exact solution and w_i the approximation.

Convergence of One-Step Methods

Theorem

Suppose the initial-value problem $y' = f(t, y)$, $a \leq t \leq b$, $y(a) = \alpha$ is approximated by a one-step difference method in the form $w_0 = \alpha$, $w_{i+1} = w_i + h\phi(t_i, w_i, h)$. Suppose also that $h_0 > 0$ exists and $\phi(t, w, h)$ is continuous with a Lipschitz condition in w with constant L on D , then

$$D = \{(t, w, h) \mid a \leq t \leq b, -\infty < w < \infty, 0 \leq h \leq h_0\}.$$

- 1 The method is stable;
- 2 The method is convergent if and only if it is consistent:

$$\phi(t, y, 0) = f(t, y)$$

- 3 If τ exists s.t. $|\tau_i(h)| \leq \tau(h)$ when $0 \leq h \leq h_0$, then

$$|y(t_i) - w_i| \leq \frac{\tau(h)}{L} e^{L(t_i - a)}.$$

Definition

Let $\lambda_1, \dots, \lambda_m$ denote the roots of the characteristic equation

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - \dots - a_1\lambda - a_0 = 0$$

associated with the multistep method

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \dots + a_0w_{i+1-m} \\ + hF(t_i, h, w_{i+1}, w_i, \dots, w_{i+1-m}).$$

If $|\lambda_i| \leq 1$ and all roots with absolute value 1 are simple, the method is said to satisfy the *root condition*.

Definition

- 1 Methods that satisfy the root condition and have $\lambda = 1$ as the only root of magnitude one are called *strongly stable*.
- 2 Methods that satisfy the root condition and have more than one distinct root with magnitude one are called *weakly stable*.
- 3 Methods that do not satisfy the root condition are *unstable*.

Theorem

A multistep method

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0w_{i+1-m} \\ + hF(t_i, h, w_{i+1}, w_i, \dots, w_{i+1-m})$$

is stable if and only if it satisfies the root condition. If it is also consistent, then it is stable if and only if it is convergent.

- A *stiff differential equation* is numerically unstable unless the step size is extremely small
- Large derivatives give error terms that are dominating the solution
- Example: The initial-value problem

$$y' = -30y, \quad 0 \leq t \leq 1.5, \quad y(0) = \frac{1}{3}$$

has exact solution $y = \frac{1}{3}e^{-30t}$. But RK4 is unstable with step size $h = 0.1$.

Euler's Method for Test Equation

- Consider the simple test equation

$$y' = \lambda y, \quad y(0) = \alpha, \quad \text{where } \lambda < 0$$

with solution $y(t) = \alpha e^{\lambda t}$.

- Euler's method gives $w_0 = \alpha$ and

$$w_{j+1} = w_j + h(\lambda w_j) = (1 + h\lambda)w_j = (1 + h\lambda)^{j+1}\alpha.$$

- The absolute error is

$$\begin{aligned} |y(t_j) - w_j| &= \left| e^{jh\lambda} - (1 + h\lambda)^j \right| |\alpha| \\ &= \left| (e^{h\lambda})^j - (1 + h\lambda)^j \right| |\alpha| \end{aligned}$$

- Stability requires $|1 + h\lambda| < 1$, or $h < 2/|\lambda|$.

Multistep Methods

Apply a multistep method to the test equation:

$$w_{j+1} = a_{m-1}w_j + \cdots + a_0w_{j+1-m} \\ + h\lambda(b_mw_{j+1} + b_{m-1}w_j + \cdots + b_0w_{j+1-m})$$

or

$$(1 - h\lambda b_m)w_{j+1} - (a_{m-1} + h\lambda b_{m-1})w_j - \cdots - (a_0 + h\lambda b_0)w_{j+1-m} = 0$$

Let β_1, \dots, β_m be the zeros of the *characteristic polynomial*

$$Q(z, h\lambda) = (1 - h\lambda b_m)z^m - (a_{m-1} + h\lambda b_{m-1})z^{m-1} - \cdots - (a_0 + h\lambda b_0)$$

Then c_1, \dots, c_m exist with

$$w_j = \sum_{k=1}^m c_k (\beta_k)^j$$

and $|\beta_k| < 1$ is required for stability.

Definition

The *region R of absolute stability* for a one-step method is $R = \{h\lambda \in \mathcal{C} \mid |Q(h\lambda)| < 1\}$, and for a multistep method, it is $R = \{h\lambda \in \mathcal{C} \mid |\beta_k| < 1, \text{ for all zeros } \beta_k \text{ of } Q(z, h\lambda)\}$.

A numerical method is said to be *A-stable* if its region R of absolute stability contains the entire left half-plane.