UCB Math 128A, Spring 2014: Programming Assignment 4 Solutions

1. To apply Newton's method on the backward Euler method

$$w_{i+1} = w_i + hf(t_{i+1}, w_{i+1}),$$

define

$$F(w_{i+1}) = w_{i+1} - w_i - hf(t_{i+1}, w_{i+1})$$

and solve $F(w_{i+1}) = 0$ using

$$w_{i+1}^{(0)} = w_i$$

$$w_{i+1}^{(k)} = w_{i+1}^{(k-1)} - \frac{F(w_{i+1}^{(k-1)})}{F'(w_{i+1}^{(k-1)})} = w_{i+1}^{(k-1)} - \frac{w_{i+1}^{(k-1)} - w_i - hf(t_{i+1}, w_{i+1}^{(k-1)})}{1 - hf_y(t_{i+1}, w_{i+1}^{(k-1)})}$$

2. The following MATLAB function implements the backward Euler method:

```
function [t,w]=backeuler(f,dfdy,a,b,alpha,N,maxiter,tol)

h=(b-a)/N;
t=(a:h:b);
w=0*t;
w(1)=alpha;
for i=1:N
  w0=w(i);
  wk=w0;
  for k=1:maxiter
    dw=(wk-w0-h*f(t(i+1),wk))/(1-h*dfdy(t(i+1),wk));
    wk=wk-dw;
    fprintf('%d %g\n',k,abs(dw));
    if abs(dw)<=tol, break; end
  end
  fprintf('\n');
  if abs(dw)>tol, error('No Newton convergence.'); end
  w(i+1)=wk;
end
```

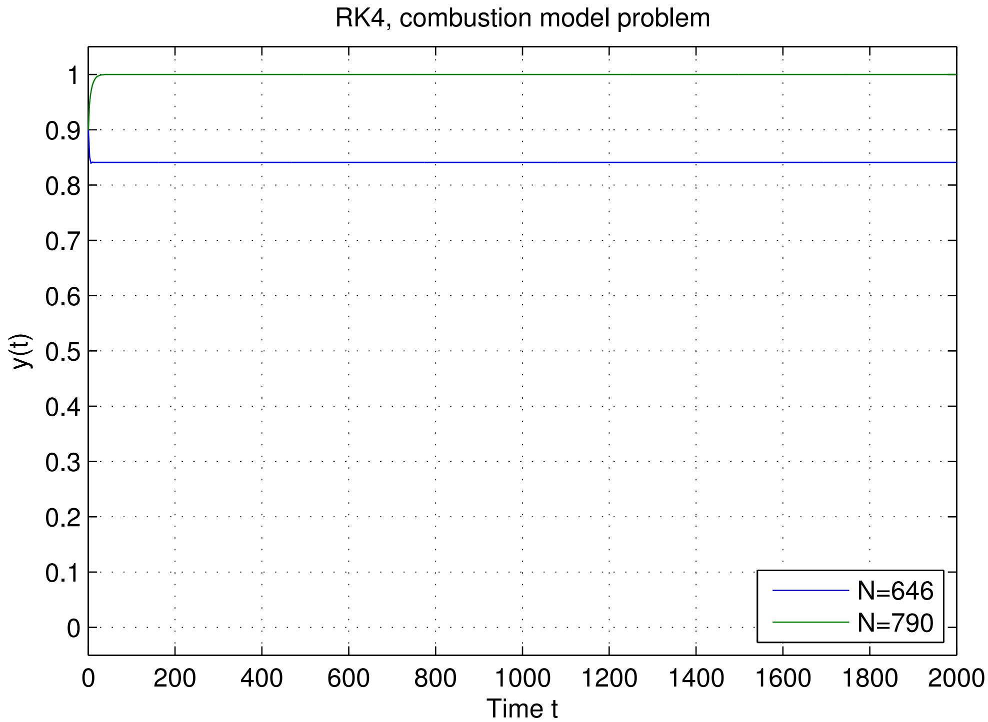
3. **a)** The step size h must approximately satisfy $|h\lambda| = h \leq 2.7853$, which gives the number of time steps $N = 2000/h \geq 718$.

b) The following lines:

```
f=@(t,y) y^2*(1-y);
dfdy=@(t,y) 2*y-3*y^2;
a=0; b=2000; alpha=0.9;

[t1,w1]=rk4(f,a,b,alpha,646);
[t2,w2]=rk4(f,a,b,alpha,790);
plot(t1,w1,t2,w2)
```

solves the problem with $N = 646 \approx 0.9 \cdot 718$ and $N = 790 \approx 1.1 \cdot 718$. The lower value gives the wrong value for $y(2000)$, but the higher gives the correct value $y(2000) = 1$.



c) Backward Euler is A-stable since if $Re(h\lambda) < 0$,

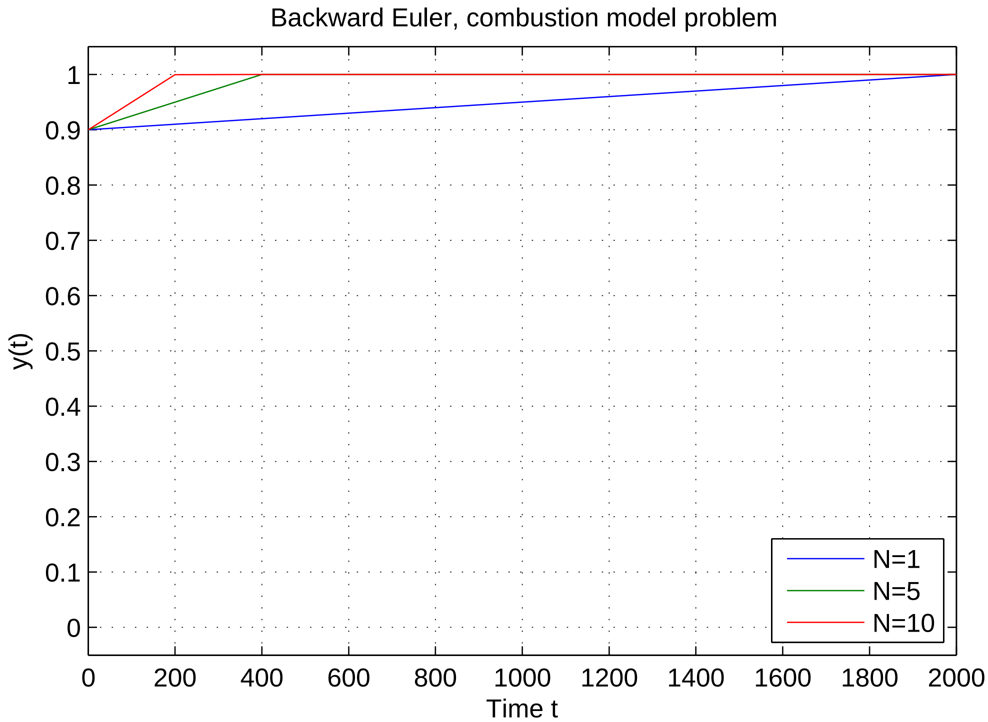
$$|Q(h\lambda)| = \left| \frac{1}{1 - h\lambda} \right| < 1.$$

This means that any N will give a stable solution, including $N = 1$.

d) The following code solves the problem for $N = 1, 5, 10$ and plots the result. All values of N appear to give stable results.

```
f=@(t,y) y^2*(1-y);
dfdy=@(t,y) 2*y-3*y^2;
a=0; b=2000; alpha=0.9; maxiter=20; tol=1e-12;

[t1,w1]=backeuler(f,dfdy,a,b,alpha,1,maxiter,tol);
[t2,w2]=backeuler(f,dfdy,a,b,alpha,5,maxiter,tol);
[t3,w3]=backeuler(f,dfdy,a,b,alpha,10,maxiter,tol);
plot(t1,w1,t2,w2,t3,w3)
```



The Newton convergence is shown below:
$N = 1$:

```
1 0.128469
2 0.0270214
3 0.00149354
4 4.46627e-006
5 3.98803e-011
6 7.88716e-018
```

$N = 5$:

```
1 0.128063
2 0.0268375
3 0.00147092
4 4.32538e-006
5 3.7348e-011
6 4.40305e-017

1 0.000249002
2 1.23664e-007
3 3.05022e-014

1 6.20646e-007
2 7.68531e-013

1 1.54774e-009
2 1.16283e-017

1 3.8597e-012
2 9.69023e-018
```

$N = 10$:

```
1 0.127559
2 0.0266097
3 0.00144316
4 4.15576e-006
5 3.44115e-011
6 1.74336e-017

1 0.000496018
2 4.89378e-007
3 4.76576e-013

1 2.46534e-006
2 1.20952e-011
3 3.50426e-017

1 1.22653e-008
2 2.5906e-016

1 6.10214e-011
2 1.76752e-017

1 3.03571e-013

1 1.54658e-015

1 0

1 0

1 0
```