Chapter 11
Boundary-Value Problems for
Ordinary Differential Equations

Per-Olof Persson
persson@berkeley.edu
Department of Mathematics
University of California, Berkeley
Math 128B Numerical Analysis

Second-order boundary-value problems

**Theorem**

Support \( f \) in the boundary-value problem

\[
y'' = f(x, y, y'), \quad \text{for } a \leq x \leq b, \text{ with } y(a) = \alpha \text{ and } y(b) = \beta
\]

is continuous on the set

\[
D = \{(x, y, y') | \text{ for } a \leq x \leq b, \text{ with } -\infty < y < \infty \text{ and } -\infty < y' < \infty \},
\]

and that the partial derivatives \( f_y, f_{y'} \) are continuous on \( D \). If

(a) \( f_y(x, y, y') > 0 \), for all \((x, y, y') \in D\), and

(ii) a constant \( M \) exists with

\[
|f_y(x, y, y')| \leq M, \quad \text{for all } (x, y, y') \in D
\]

then the boundary-value problem has a unique solution.

Linear boundary-value problems

The differential equation \( y'' = f(x, y, y') \) is linear when functions \( p(x), q(x), \) and \( r(x) \) exist with

\[
f(x, y, y') = p(x)y' + q(x)y + r(x)
\]

**Corollary**

Suppose the linear boundary-value problem

\[
y'' = p(x)y' + q(x)y + r(x), \quad \text{for } a \leq x \leq b,
\]

with \( y(a) = \alpha \) and \( y(b) = \beta \), satisfies

(i) \( p(x), q(x), \) and \( r(x) \) are continuous on \([a, b]\)

(ii) \( q(x) > 0 \) on \([a, b]\)

Then the boundary-value problem has a unique solution.

The shooting method for nonlinear problems

For the general problem

\[
y'' = f(x, y, y'), \quad \text{with } a \leq x \leq b, \quad \text{and } \quad y(a) = \alpha, \quad y(b) = \beta
\]

consider the sequence of initial-value problems, parameterized by \( t \)

\[
y'' = f(x, y, y'), \quad \text{with } a \leq x \leq b, \quad y(a) = \alpha, \quad y'(a) = t
\]

with \( t \) chosen such that

\[
\lim_{k \to \infty} y(b, t_k) = y(b) = \beta
\]

Choose \( t_k \) as solution to the nonlinear equation

\[
y(b, t) - \beta = 0
\]

For example, using the Secant method, with initial approximations \( t_0, t_1 \) the remaining terms are given by

\[
t_k = t_{k-1} - \frac{y(b, t_{k-1}) - \beta}{y(b, t_{k-1}) - y(b, t_{k-2})}, \quad k = 2, 3, \ldots
\]

The linear shooting method

- Consider the linear boundary-value problem

\[
y'' = p(x)y' + q(x)y + r(x), \quad \text{with } a \leq x \leq b, \quad y(a) = \alpha, \quad y(b) = \beta
\]

- In the linear shooting method, split into two problems

\[
y'' = p(x)y' + q(x)y + r(x), \quad \text{with } a \leq x \leq b, \quad y(a) = \alpha, \quad y'(a) = 0
\]

\[
y'' = p(x)y' + q(x)y, \quad \text{with } a \leq x \leq b, \quad y(a) = 0, \quad y'(a) = 1
\]

with solutions denoted \( y_1(x) \), \( y_2(x) \), respectively

- The solution to the original boundary-value problem is then

\[
y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2(x)
\]

- Two initial value problems \( \Rightarrow \) solve numerically using e.g. the fourth-order Runge-Kutta method

Shooting method with Newton iterations

To use Newton's method for \( \{t_k\} \), start from initial approximation \( t_0 \) and iterate

\[
t_k = t_{k-1} - \frac{y(b, t_{k-1}) - \beta}{\frac{dy}{dt}(b, t_{k-1})}
\]

To determine \( (dy/dt)(b, t) \), solve another differential equation for

\[
z(x, t) = (\partial y/\partial t)(x, t):
\]

\[
z''(x, t) = \frac{\partial f}{\partial y}(x, y, y') z(x, t) + \frac{\partial f}{\partial y'}(x, y, y') z'(x, t), \quad \text{for } a \leq x \leq b
\]

with \( z(a, t) = 0 \) and \( z(a, t) = 1 \), and use the Newton iteration

\[
t_k = t_{k-1} - \frac{y(b, t_{k-1}) - \beta}{z(b, t_{k-1})}
\]
Finite-difference methods for linear problems

- Introduce a mesh for the interval \([a, b]\), by choosing an integer \(N > 0\), set \(h = (b - a)/(N + 1)\) and \(x_i = a + ih\), for \(i = 0, 1, \ldots, N + 1\).
- Use a centered-difference formula for \(y''(x_i)\):
  \[
y''(x_i) = \frac{1}{h^2}[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})] - \frac{h^2}{12}y^{(4)}(\xi_i)
  \]
  and for \(y'(x)\):
  \[
y'(x_i) = \frac{1}{h}[y(x_{i+1}) - y(x_{i-1})] - \frac{h}{6}y''(\eta_i)
  \]
- Use these difference formulas to approximate the differential equation at the interior mesh points:
  \[
y''(x_i) = p(x_i)y'(x_i) + q(x_i)y(x_i) + r(x_i)
  \]

Finite-difference methods for nonlinear problems

- For a general nonlinear boundary-value problem
  \[
y'' = f(x, y, y'), \quad \text{for} \quad a \leq x \leq b, \quad \text{with} \quad y(a) = \alpha \quad \text{and} \quad y(b) = \beta
  \]
  replace each derivative with finite differences at points \(x_i\), \(i = 1, \ldots, N\), like before:
  \[
y''(x_i) = f(x_i, y(x_i), y'(x_i))
  \]
  This leads to the equations
  \[
  -\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} + f \left( x_i, w_i, \frac{w_{i+1} - w_{i-1}}{2h} \right) = 0,
  \]
  for \(i = 1, 2, \ldots, N\), with the boundary conditions
  \[
w_0 = \alpha, \quad w_{N+1} = \beta
  \]
  - Nonlinear system of equations \(\Rightarrow\) use Newton’s method

Nonlinear problems, conditions for unique solution

For the nonlinear problem
\[
y'' = f(x, y, y'), \quad \text{for} \quad a \leq x \leq b, \quad \text{with} \quad y(a) = \alpha \quad \text{and} \quad y(b) = \beta
\]
if \(f\) satisfies:
- \(f, f_y, f_{y'}\) are continuous on
  \[
  D = \{(x, y, y') \mid a \leq x \leq b, \quad \text{with} \quad -\infty < y < \infty \quad \text{and} \quad -\infty < y' < \infty\}
  \]
- \(f_y(x, y, y') \geq \delta\) on \(D\), for some \(\delta > 0\)
- Constants \(k, L\) exists with
  \[
  k = \max_{(x, y, y') \in D} |f_y(x, y, y')| \quad \text{and} \quad L = \max_{(x, y, y') \in D} |f_{y'}(x, y, y')|
  \]
then a unique solution exists

The Rayleigh-Ritz Method

- Consider the linear boundary-value problem
  \[
  -\frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) + q(x)y = f(x), \quad \text{for} \quad 0 \leq x \leq 1
  \]
  with \(y(0) = y(1) = 0\).
- Assume \(p \in C^1[0, 1]\) and \(q, f \in C[0, 1]\), and that a constant \(\delta > 0\) exists such that
  \[
p(x) \geq \delta, \quad q(x) \geq 0, \quad \text{for each} \quad x \in [0, 1]
  \]
- Then the problem has a unique solution.
**Variational Problem**

**Theorem**

Let \( p \in C^1[0,1] \), \( q, f \in C[0,1] \), and 
\[
p(x) \geq \delta > 0, \quad q(x) \geq 0, \quad \text{for } 0 \leq x \leq 1.
\]

Then \( y \in C^2_0[0,1] \) is the unique solution to 
\[
\frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) q(x)y = f(x), \quad \text{for } 0 \leq x \leq 1
\]
if and only if \( y \) is the unique function in \( C^2_0[0,1] \) that minimizes 
\[
I[u] = \int_0^1 \left\{ p(x)[u'(x)]^2 + q(x)[u(x)]^2 - 2f(x)y(x) \right\} dx
\]

**Variational Problem**

- The Rayleigh-Ritz method is based on the variational minimization property, but only minimizing over the set of linear combinations of certain basis functions \( \phi_1, \ldots, \phi_n \).
- \( \{\phi_i\} \) are linearly independent, and \( \phi_i(0) = \phi_i(1) = 0 \), for \( i = 1, \ldots, n \).
- The minimization problem then leads to the normal equations 
  \[
  Ac = b, \quad \text{where}
  a_{ij} = \int_0^1 \left\{ p(x)\phi_i'(x)\phi_j'(x) + q(x)\phi_i(x)\phi_j(x) \right\} dx \\
  b_i = \int_0^1 f(x)\phi_i(x) dx
  \]

**Piecewise-Linear Basis**

Choose piecewise-linear polynomials for basis functions: Partition \([0,1]\) by points:
\[0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1\]

Set \( h_i = x_{i+1} - x_i \), and define \( \phi_i(x) \) by 
\[
\phi_i(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq x_{i-1} \\
\frac{1}{h_{i-1}}(x - x_{i-1}) & \text{if } x_{i-1} < x \leq x_i \\
\frac{1}{h_i}(x_{i+1} - x) & \text{if } x_i < x \leq x_{i+1} \\
0 & \text{if } x_{i+1} < x \leq 1
\end{cases}
\]

with derivatives 
\[
\phi'_i(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq x_{i-1} \\
\frac{1}{h_{i-1}} & \text{if } x_{i-1} < x \leq x_i \\
-\frac{1}{h_i} & \text{if } x_i < x \leq x_{i+1} \\
0 & \text{if } x_{i+1} < x \leq 1
\end{cases}
\]

Note: \( \phi_i(x)\phi_j(x) = \phi'_i(x)\phi'_j(x) = 0 \) if \( j < i - 1 \) or \( j > i + 1 \).