**Lecture 14**

**Hessenberg/Tridiagonal Reduction**

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Introduction to Numerical Methods

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**Introducing Zeros by Similarity Transformations**

- Try computing the Schur factorization $A = Q T Q^*$ by applying Householder reflectors from left and right that introduce zeros:

- The right multiplication destroys the zeros previously introduced
- We already knew this would not work, because of Abel’s theorem
- However, the subdiagonal entries typically decrease in magnitude

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**The Hessenberg Form**

- Instead, try computing an upper Hessenberg matrix $H$ similar to $A$:

- This time the zeros we introduce are not destroyed
- Continue in a similar way with column 2:

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**Householder Reduction to Hessenberg**

**Algorithm: Householder Hessenberg**

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for k = 1 to m - 2
  x = A_{k+1,m,k}
  v_k = sign(x_i)||x|| e_1 + x
  v_k = v_k/||v_k||
  A_{k+1,m,k,m} = A_{k+1,m,k,m} - 2 v_k (v_k^T A_{k+1,m,k,m})
  A_{1,m,k+1,m} = A_{1,m,k+1,m} - 2 (A_{1,m,k+1,m} v_k^T v_k)
```

- Operation count (not twice Householder QR):

  $\sum_{k=1}^m 4(m-k)^2 + 4m(m-k) = 4m^3/3 + 4m^3 - 4m^3/2 = 10m^3/3$

- For hermitian $A$, operation count is twice QR divided by two $= 4m^3/3$

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**Stability of Householder Hessenberg**

- The Householder Hessenberg reduction algorithm is backward stable:

  $\tilde{Q} H \tilde{Q}^* = A + \delta A$, $\frac{||\delta A||}{||A||} = O(\epsilon_{machine})$

  where $\tilde{Q}$ is an exactly unitary matrix based on $\tilde{v}_k$