MIT 18.335, Fall 2007: Midterm

November 7, 2007

Name: ________________________________

- Do all of the 8 problems
- Justify your answers
- Exam time 90 minutes

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1. (10 points)

(a) Give the definition of induced matrix norms.

(b) Prove the inequality $\|AB\| \leq \|A\| \cdot \|B\|$ for induced matrix norms.

(c) Suppose $A$ has right singular vectors $v_i^A$ and $B$ has left singular vectors $u_i^B$. Give a sufficient condition on these vectors to achieve the equality $\|AB\|_2 = \|A\|_2 \cdot \|B\|_2$. 
2. (10 points)

(a) Give the definition of a projector.

(b) *Onto* which space does a projector $P$ project, and *along* which space does it project? Give brief motivations.

(c) Find the projector $P$ that projects onto the space $\{(2, 1)\}$ along the space $\{(3, -1)\}$. 
3. (10 points)

Use Gram-Schmidt orthogonalization to find a reduced QR factorization of the matrix

\[
A = \begin{bmatrix}
0 & \sqrt{2} \\
1 & 4 \\
1 & 2
\end{bmatrix}
\]
4. (10 points)

In many algorithms, we use Householder reflectors of the form

\[ F = I - 2 \frac{vv^*}{v^*v}, \quad \text{where} \quad v = \text{sign}(x_1)\|x\|e_1 + x. \]

(a) Show algebraically that \( F \) is unitary.

(b) What are the eigenvalues of \( F \)?

(c) Describe with a simple drawing the effect of applying the reflector \( F \) on \( x \), and explain the choice of sign in the expression for \( v \).
5. (15 points)

The following algorithm is implemented on a computer satisfying the two usual floating point axioms:

Data $x_1, x_2 \in \mathbb{C}$, solution $(x_1 + 1)/x_2$, computed as $(\text{fl}(x_1) \oplus 1) \odot \text{fl}(x_2)$.

State if it is backward stable, stable but not backward stable, or unstable, and show why. For perturbed data, determine the coefficients of first order terms, such as in “$|\epsilon| \leq k\epsilon_{\text{machine}} + O(\epsilon_{\text{machine}}^2)$”.
6. (15 points)

For each of the following statements, determine if it is true or false, and justify your answer. \( A \in \mathbb{C}^{m \times m} \).

(a) If \( v \) and \( w \) are linearly independent, \( A = vv^* + ww^* \) has rank 2.

(b) In MATLAB, \( \left(1/0\right) + \left(1/(-0)\right) \) returns NaN.

(c) If \( A \) has upper and lower bandwidth \( p \) and it has an LU factorization with partial pivoting \( PA = LU \), then \( U \) has upper bandwidth \( p \).

(d) If \( A \) is hermitian, it has a Cholesky factorization \( A = R^*R \).

(e) If all eigenvalues of \( A \) are distinct, it has an eigenvalue decomposition \( A = X\Lambda X^{-1} \).
Consider the following algorithm applied to an $m$-by-$m$ matrix $A$:

$$B = A$$

for $k = 1$ to $m$

$$x = B_{k:m,k}$$
$$v_k = \text{sign}(x_1)\|x\|_2 e_1 + x$$
$$v_k = v_k / \|v_k\|_2$$
$$B_{k:m,k:m} = B_{k:m,k:m} - 2v_k (v_k^* B_{k:m,k:m})$$
$$x = B_{k,k+1:m}^T$$
$$w_k = \text{sign}(x_1)\|x\|_2 e_1 + x$$
$$w_k = w_k / \|w_k\|_2$$
$$B_{k:m,k+1:m} = B_{k:m,k+1:m} - 2(B_{k:m,k+1:m} w_k) w_k^*$$

(a) What does the algorithm do, and what is the structure of $B$?

(b) How are the singular values of $B$ related to those of $A$?

(c) Give the operation count of the algorithm (leading term only).
8. (15 points)

(a) Describe an iterative algorithm (in pseudo-code) for computing $\|A\|_2$ for an $m$-by-$n$ matrix $A$, which only uses $A$ in the form of matrix-vector products $Ax$ and $A^*y$.

(b) State under which conditions your algorithm converges, and at which rate (in terms of appropriate properties of $A$).