1 - Integration by Parts

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Math 1B Calculus

Integrals

Indefinite Integrals

$$\int f(x)\,dx = F(x) \quad \text{means} \quad F'(x) = f(x)$$

Result is *a function* (or a family of functions)

Definite Integrals, Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$$

Result is *a number*

The Substitution Rule

The Substitution Rule

$$\int f(g(x))g'(x)\,dx = \left[\begin{array}{ccc} u & = & g(x) \\ du & = & g'(x)\,dx \end{array}\right] = \int f(u)\,du$$

Example

$$\int 2xe^{x^2}\,dx = \left[\begin{array}{ccc} u & = & x^2 \\ du & = & 2x\,dx \end{array}\right] = \int e^u\,du = e^u + C = e^{x^2} + C$$

Integration by Parts

Integration by Parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Proof.

Recall product rule for *differentiation*:

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$$

Integrate:

$$f(x)g(x) = \int \left[f(x)g'(x) + g(x)f'(x)\right]\,dx$$

Rearrange:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Integration by Parts

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$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Alternative Notation

Let $u = f(x)$ and $v = g(x)$.
Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$.
The integration by parts formula becomes:

$$\int u\,dv = uv - \int v\,du$$

Integration by Parts

Example

$$\int xe^x\,dx = \left[\begin{array}{cccccc} f(x) & = & x, & f'(x) & = & 1 \\ g'(x) & = & e^x, & g(x) & = & e^x \end{array}\right]$$

$$= xe^x - \int e^x\,dx = xe^x - e^x + C = e^x(x-1) + C$$

Example

$$\int \ln x\,dx = \left[\begin{array}{ccc} f(x) & = & \ln x \\ g'(x) & = & 1 \end{array}\right] = x\ln x - \int x \cdot \frac{1}{x}\,dx$$

$$= x\ln x - \int 1 \cdot dx = x\ln x - x + C$$

How to Choose $f(x)$ and $g'(x)$

Integration by Parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

- Must be able to integrate $g'(x)$
- Must be able to integrate $g(x)f'(x)$

LIATE rule

Choose $f(x)$ as high up as possible:

- Logarithmic (e.g. $\ln x$)
- Inverse Trigonometric (e.g. $\sin^{-1} x$)
- Algebraic (e.g. x^{-3})
- Trigonometric (e.g. $\sin x$)
- Exponential (e.g. e^x)

Definite Integrals

Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b g(x)f'(x)\,dx$$

Example

$$\int_4^9 \frac{\ln y}{\sqrt{y}}\,dy = \left[\begin{array}{rclrcl} f(y) & = & \ln y, & f'(y) & = & \frac{1}{y} \\ g'(y) & = & \frac{1}{\sqrt{y}}, & g(y) & = & 2\sqrt{y} \end{array}\right]$$

$$= [\ln y \cdot 2\sqrt{y}]_4^9 - \int_4^9 2\sqrt{y}\frac{1}{y}\,dy$$

$$= 6\ln 9 - 4\ln 4 - \int_4^9 \frac{2}{\sqrt{y}}\,dy$$

$$= 6\ln 9 - 4\ln 4 - [4\sqrt{y}]_4^9 = 6\ln 9 - 4\ln 4 - 4$$