5 - Integration of Rational Functions by Partial Fractions

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Math 1B Calculus

Partial Fractions

Case 3: Distinct Irreducible Quadratic Factors

$$Q(x) = (ax^2 + bx + c) \cdots \Longrightarrow \frac{R(x)}{Q(x)} = \frac{Ax + B}{ax^2 + bx + c} + \cdots$$

Example

$$\begin{aligned}\frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx + D}{x^2 + 1} \\ x^2 - 2x - 1 &= A(x-1)(x^2+1) + B(x^2+1) + \\ &\quad (Cx + D)(x-1)^2 \\ &= (A + C)x^3 + (-A + B - 2C + D)x^2 + \\ &\quad (A + C - 2D)x + (-A + B + D)\end{aligned}$$

Partial Fractions

Example

$$\begin{cases} A + C & = & 0 \\ -A + B - 2C + D & = & 1 \\ A + C - 2D & = & -2 \\ -A + B + D & = & -1 \end{cases}$$

Solve by elimination $\Longrightarrow A = 1, B = -1, C = -1, D = 1 \Longrightarrow$

$$\int \frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)}\,dx = \int \frac{1}{x-1} - \frac{1}{(x-1)^2} - \frac{x-1}{x^2+1}\,dx$$

$$= \ln|x-1| + \frac{1}{x-1} - \frac{1}{2}\ln(x^2+1) + \tan^{-1} x + C$$

Integrating Irreducible Quadratic Partial Fractions

How to integrate $\dfrac{Ax+B}{ax^2+bx+c}$

1) Complete the square and substitute $\Longrightarrow ax^2+bx+c = u^2+d$
2) Split numerator into a term with u and a constant term
3) Integration of first term gives $\ln$, second term gives $\tan^{-1}$

Example

$$\int \frac{x+4}{x^2-4x+6}\,dx = \int \frac{x+4}{(x-2)^2+2}\,dx = \begin{bmatrix} u & = & x-2 \\ du & = & dx \end{bmatrix}$$

$$= \int \frac{u+6}{u^2+2}\,du = \int \frac{u}{u^2+2}\,du + \int \frac{6}{u^2+2}\,du$$

$$= \frac{1}{2}\ln(u^2+2) + 6\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}} + C$$

$$= \frac{1}{2}\ln(x^2-4x+6) + 3\sqrt{2}\tan^{-1}\frac{x-2}{\sqrt{2}} + C$$

Partial Fractions

Case 4: Repeated Irreducible Quadratic Factors

$$Q(x) = (ax^2 + bx + c)^r \cdots \Longrightarrow \frac{R(x)}{Q(x)} = \frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_r x + B_r}{(ax^2 + bx + c)^r} + \cdots$$

Example

$$\frac{2x+1}{(x+1)^3(x^2+4)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$$

Integrating Squared Quadratic Fractions

$$\int \frac{1}{(x^2+a^2)^2}\,dx = \left[\begin{array}{rclrcl} x &=& a\tan\theta, & \sin\theta &=& \frac{x}{\sqrt{x^2+a^2}} \\ dx &=& a\sec^2\theta\,d\theta, & \cos\theta &=& \frac{a}{\sqrt{x^2+a^2}} \end{array}\right]$$

$$= \int \frac{1}{a^4\sec^4\theta}\cdot a\sec^2\theta\,d\theta = \frac{1}{a^3}\int \cos^2\theta\,d\theta$$

$$= \frac{1}{2a^3}\int (1+\cos 2\theta)\,d\theta = \frac{\theta}{2a^3} + \frac{\sin 2\theta}{4a^3} + C$$

$$= \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + \frac{\frac{x}{\sqrt{x^2+a^2}}\cdot\frac{a}{\sqrt{x^2+a^2}}}{2a^3} + C$$

$$= \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + \frac{x}{2a^2(x^2+a^2)} + C$$