Your Name: ______________________________

1a *(3 points)*. Express as a definite integral the length of the curve $y = \sin x$, $0 \le x \le \pi/2$.

$$\text{Length} = \int_0^{\pi/2} \sqrt{1 + \cos^2 x}\, dx$$

1b *(5 points)*. Decide whether $\sum_{n=1}^{\infty} (-1)^n \frac{2^n n!}{5 \cdot 8 \cdot 11 \cdots (3n+2)}$ converges absolutely, converges conditionally, or diverges.

By Ratio Test,

$$\lim_{n \to \infty} \left| \frac{(-1)^{n+1}\, 2^{n+1}\, (n+1)!}{5 \cdot 8 \cdot 11 \cdots (3n+5)} \cdot \frac{5 \cdot 8 \cdot 11 \cdots (3n+2)}{(-1)^n\, 2^n\, n!} \right| =$$

$$= \lim_{n \to \infty} \frac{2\,(n+1)}{3n+5} = \lim_{n \to \infty} \frac{2\left(1 + \frac{1}{n}\right)}{3 + \frac{5}{n}} = \frac{2}{3} < 1$$

Thus, the series is absolutely convergent.

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2a *(4 points)*. Find $f^{(666)}(0)$ if $f(x) = \sec(x^{333})$.

The first few terms of Maclaurin series are easy to find for $\sec x$:

So, $\sec x = 1 + \frac{1}{2}x^2 + \cdots$

Or, $\sec(x^{333}) = 1 + \frac{1}{2}x^{666} + \cdots$

Since $f(x) = \sec(x^{333})$, we must have

$$\frac{f^{(666)}(0)}{(666)!} = \frac{1}{2}$$ to be the coefficient of x^{666} in the power series of $f(x)$.

Or, $$f^{(666)}(0) = \frac{(666)!}{2}$$

2b *(4 points)*. A tank contains 500 L of pure water. Brine that contains 0.05 kg of salt per liter flows into the tank at the rate of 8 L/min. The solution is kept thoroughly mixed and drains from the tank at the rate of 8 L/min. Let $y(t)$ be the amount of salt in the solution at time t, measured in kg; time is measured in minutes. What differential equation is satisfied by y? Without solving this equation, guess the value of $\lim_{t\to\infty} y(t)$.

$$\frac{dy}{dt} = \text{rate in} - \text{rate out}$$

$$\text{rate in} = \left(0.05\ \frac{kg}{L}\right)\cdot\left(8\ \frac{L}{min}\right) = 0.4\ \frac{kg}{min}$$

$$\text{rate out} = \left(\frac{y(t)}{500}\ \frac{kg}{L}\right)\cdot\left(8\ \frac{L}{min}\right) = \frac{y(t)}{62.5}\ \frac{kg}{min}$$

So, the differential equation is:

$$\frac{dy}{dt} = 0.4 - \frac{y(t)}{62.5} = \frac{25 - y(t)}{62.5}$$

The amount of salt will approach to 25, since it will always increase and have upper bound when the derivative is zero.

i.e. $\lim_{t\to\infty} y(t) = 25$

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3a *(4 points)*. Find one solution to $y'' + 5y' + 6y = \cos x$.

Since $G(x) = \cos x$, we can try $y_p(x) = a\cdot\cos x + b\sin x$ as a particular solution.

We have $y_p'(x) = -a\sin x + b\cos x$

$y_p''(x) = -a\cos x - b\sin x$

Putting back into differential equation, we get

$$-a\cdot\cos x - b\cdot\sin x - 5a\sin x + 5b\cos x + 6a\cos x + 6b\sin x = \cos x$$

or, $(-a+5b+6a)\cos x + (-b-5a+6b)\sin x = \cos x$

or $\begin{cases} -a+5b+6a = 1 \\ -b-5a+6b = 0 \end{cases}$ | or $\begin{cases} 5a+5b = 1 \\ -5a+5b = 0 \end{cases}$ | or $\begin{cases} 10a = 1 \\ a = b \end{cases}$ | $a = b = \frac{1}{10}$

So, one solution is $y(x) = \frac{1}{10}\cos x + \frac{1}{10}\sin x$

3b *(5 points)*. Use power series methods to solve the initial-value problem:
$y'' + xy' - 2y = 0$, $y(0) = 1$, $y'(0) = 0$.

We assume the solution is in the form $y = \sum_{n=0}^{\infty} c_n x^n$

Then, $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$.

We substitute these into differential equation to get:

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + x\sum_{n=1}^{\infty} n c_n x^{n-1} - 2\sum_{n=0}^{\infty} c_n x^n = 0$$

or, $\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + \sum_{n=0}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} 2c_n x^n = 0$

or, $\sum_{n=0}^{\infty}\left[(n+2)(n+1)c_{n+2} + nc_n - 2c_n\right]x^n = 0$, or $(n+2)(n+1)c_{n+2} + (n-2)c_n = 0$

or $c_{n+2} = -\frac{(n-2)}{(n+2)(n+1)}c_n$

We calculate few first terms:

$\boxed{n=0}$ $c_2 = \frac{2}{2}c_0 = c_0$

$\boxed{n=1}$ $c_3 = \frac{1}{2\cdot 3}c_1 = \frac{1}{6}c_1$

$\boxed{n=2}$ $c_4 = 0$

$\boxed{n=3}$ $c_5 = -\frac{1}{5\cdot 4}c_3 = -\frac{1}{5\cdot 4}\cdot\frac{1}{2\cdot 3}c_1 = -\frac{1}{5!}c_1$

$\boxed{n=4}$ $c_6 = -\frac{2}{6\cdot 5}c_4 = 0$

Note all even terms after c_2 are 0.

$\boxed{n=5}$ $c_7 = -\frac{3}{7\cdot 6}\cdot c_5 = \frac{3}{7\cdot 6}\frac{1}{5\cdot 4}\cdot\frac{1}{2\cdot 3}\cdot c_1 = \frac{3}{7!}c_1$

We find that $c_{2n} = 0$ and $c_{2n+1} = \frac{(-1)^{n+1}(2n-3)\cdots 7\cdot 5\cdot 3\cdot 1}{(2n+1)!}c_1$

We get $y = c_0 + \sum_{n=1}^{\infty} c_{2n+1}x^{2n+1} + c_1 x + c_0 x^2$

$y(0) = c_0 = 1 \Rightarrow c_0 = 1$

$y' = c_1 + \sum_{n=1}^{\infty}(2n+1)c_{2n+1}x^{2n} + 2c_0 x$

$y'(0) = c_1 = 0$

$\boxed{y = 1 + x^2}$ 4

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4a) $y = f(x)$ and $y' = y^2 + 2y$, $y(2) = 1$.

Thus, $y'(2) = (y(2))^2 + 2y(2)$

$= 1 + 2.1 = 3. \; (= f'(2))$.

<u>So:</u> to find $y''(2) \; (= f''(2))$ we differentiate;

$y' = y^2 + 2y$, to obtain

$y'' = 2yy' + 2y'$

$\Rightarrow y''(2) = 2y(2)y'(2) + 2y'(2)$

$= 2.1.3 + 2.3 = 12.$

b) We have the family of curves

$\{2x^2 + y^2 = k \mid k \text{ is some number}\}$.

The orthogonal curve to this family passing through $(1,1)$, call it C, must have $\frac{dC}{dx} \cdot \frac{dy}{dx} = -1$, where we have $\frac{dy}{dx}$ obtained as follows: use the equation for the family of curves

$2x^2 + y^2 = k$

Differentiating
$\Rightarrow \quad 4x + 2y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2x}{y}$

$\Rightarrow \frac{dC}{dx} = \frac{-1}{-\frac{2x}{y}} = \frac{y}{2x}$

But $y = C(x)$ so we have

$$\frac{dC}{C} = \frac{dx}{2x}$$

integrate

$$\Rightarrow \quad \ln|C| = \frac{1}{2}\ln x + c$$

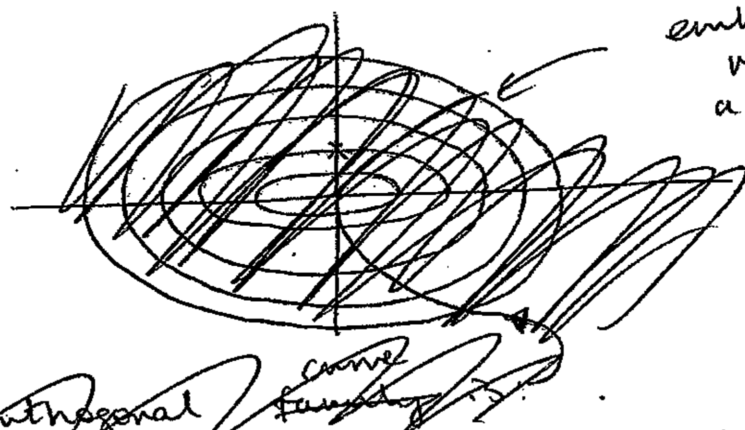
$$= \cancel{\tfrac{1}{2}} \ln e^{c}|x|^{1/2}$$

$$\Rightarrow \quad |C| = A|x|^{1/2},$$

or $\quad C^2 = Ax$

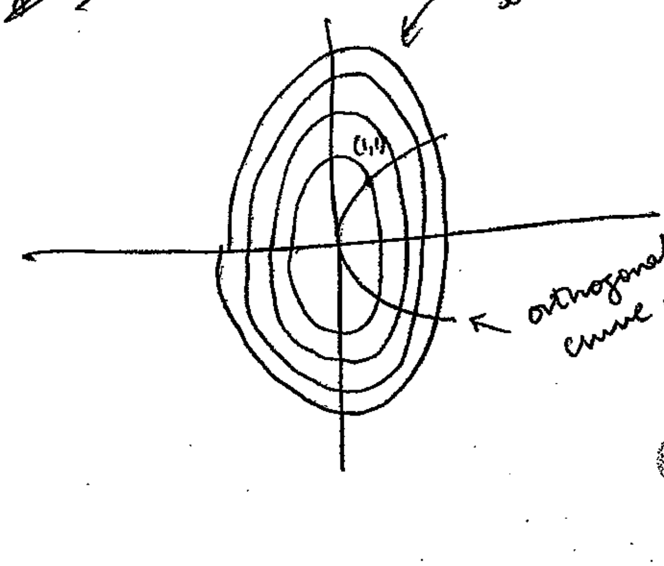
$\underline{ie}$ if $y = C(x)$, $\; y^2 = Ax$, as this curve passes through $(1,1)$, we have $1 = A \Rightarrow y^2 = x$ is the desired orthogonal curve

In pictures: the family $2x^2 + y^2 = k$ are ellipses



each curve represents a different k.

~~the orthogonal curve family is~~



5a) $$\int_0^2 \sqrt{2x-x^2}\,dx$$

$$= \int_0^2 \sqrt{x(2-x)}\,dx \quad , \quad \text{this is well-defined on } [0,2].$$

Now, $$2x - x^2 = 1 - 1 + 2x - x^2$$
$$= 1 - (1 - 2x + x^2)$$
$$= 1 - (1-x)^2$$

So, let $u = 1-x$, $du = -dx$

$$\Rightarrow \quad -\int_1^{-1} \sqrt{1-u^2}\,du = \int_{-1}^{1} \sqrt{1-u^2}\,du$$

Let $u = \sin t$, $-\frac{\pi}{2} \leq t \leq \pi/2$

$du = \cos t$

$$= \int_{-\pi/2}^{\pi/2} |\cos t| \cos t\,dt = \int_{-\pi/2}^{\pi/2} \cos^2 t\,dt \quad , \text{ since}$$

where we have used $\sqrt{1-\sin^2 t} = |\cos t|$,

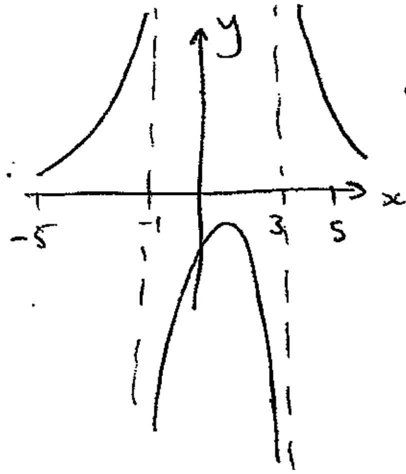
and $\cos t \geq 0$ for $-\frac{\pi}{2} \leq t \leq \pi/2$.

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{2}(\cos 2t + 1)\,dt = \frac{1}{2}\left[\frac{\sin 2t}{2} + t\right]_{-\pi/2}^{\pi/2}$$
$$= \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{\pi}{2}.$$

6) $\displaystyle\int_{-5}^{5} \frac{4\,dt}{t^2-2t-3}$. Note; t^2-2t-3
$= (t-3)(t+1)$

So there discontinuities at
$t=-1$ and $t=3$.

The graph is something like



$$y = \frac{4}{x^2-2x-3}$$

So: $$\int_{-5}^{5} \frac{4\,dt}{t^2-2t-3} = \lim_{s\to -1^-}\int_{-5}^{s} \frac{4\,dt}{(t-3)(t+1)} + \lim_{r\to -1^+}\int_{r}^{0}\frac{4\,dt}{(t-3)(t+1)}$$

$$+ \lim_{p\to 3^-}\int_{0}^{p}\frac{4\,dt}{(t-3)(t+1)}$$

$$+ \lim_{q\to 3^+}\int_{q}^{5}\frac{4\,dt}{(t-3)(t+1)}$$

Now: $$\frac{4}{(t-3)(t+1)} = \frac{A}{t-3} + \frac{B}{t+1} = \frac{(A+B)t + A - 3B}{(t-3)(t+1)}$$

$\Rightarrow$ $A+B=0$, $A-3B=4$ $\Rightarrow$ $A=1$, $B=-1$

$$\Rightarrow \int \frac{4\,dt}{(t-3)(t+1)} = \int \left(\frac{1}{t-3} - \frac{1}{t+1}\right)dt$$

$$= \ln|t-3| - \ln|t+1| \quad (+C).$$

$$= \ln\left|\frac{t-3}{t+1}\right|.$$

As $\displaystyle\lim_{t\to -1^+} \ln\left|\frac{t-3}{t+1}\right|$ is undefined (it's ∞), the ~~indefinite~~ improper integral does not exist.

c) $\displaystyle\int \cos\sqrt{x}\,dx$. Let $u = \sqrt{x}$

$du = \frac{1}{2}\frac{1}{\sqrt{x}}\,dx$

$\Rightarrow \quad 2u\,du = dx$

$$= 2\int u\cos u\,du$$

Use integration by parts:

$w = u \qquad dz = \cos u\,du$

$dw = du \qquad z = \sin u$

$$= 2\left(u\sin u - \int \sin u\,du\right)$$

$$= 2u\sin u + 2\cos u + C.$$

$$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

d) $\displaystyle\int_e^{\infty} \frac{dx}{x(\ln x)^2}$. Let $u = \ln x$

$du = \frac{dx}{x}$

$$= \int_1^{\infty} \frac{du}{u^2} = \lim_{t\to\infty}\int_1^{t} \frac{du}{u^2}$$

$$= \lim_{t\to\infty}\left[-u^{-1}\right]_1^t$$

$$= \lim_{t\to\infty}\left(-\frac{1}{t} + 1\right) = 1.$$

6a. Test for convergence:

$$\sum_{n=0}^{\infty} \sqrt{\frac{2^n + 3^n}{6^n}}.$$

Solution. We have

$$\sqrt{\frac{2^n + 3^n}{6^n}} \leq \sqrt{\frac{3^n + 3^n}{6^n}} = \left(\frac{1}{\sqrt{2}}\right)^{n-1}.$$

The series $\sum_{n=0}^{\infty}(1/\sqrt{2})^{n-1}$ is geometric and its common ratio is less than one in magnitude, so it converges. Therefore by the comparison test the original series converges as well.

6b. How many terms of the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$$

are required to approximate the actual sum with an error of less than 5×10^{-4}?

Solution. Let $b_n = 1/(2n)!$. This is decreasing and approaches a limit of zero, so we can apply the alternating series estimation theorem to conclude that the error in approximating the sum by the first n terms $-b_1 + b_2 - \cdots \pm b_n$ is $\leq b_{n+1}$. So it suffices to find n such that $b_{n+1} = 1/(2n+2)! < 5 \times 10^{-4}$, or equivalently, $(2n+2)! > 2000$. We have $6! = 720$, and clearly $8! = 8 \cdot 7 \cdot 6! > 2000$, so three terms suffice.

7a. Using a series representation for $(1+x)^{1/2}$, evaluate

$$\sum_{n=2}^{\infty}(-1)^n \frac{(1\cdot 3\cdot 5\cdots(2n-3))9^n}{32^n n!}.$$

Solution. Using the binomial series expansion, we have

$$\begin{aligned}(1+x)^{1/2} &= \sum_{n=0}^{\infty}\binom{1/2}{n}x^n \\ &= \sum_{n=0}^{\infty}\frac{(1/2)(-1/2)(-3/2)(-5/2)\cdots((3-2n)/2)}{n!}x^n \\ &= 1+\frac{1}{2}x+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!}x^n.\end{aligned}$$

If $x = 9/16$ then the sum of the series in the third term is $-S$, where S is the sum we want to evaluate. Therefore

$$S = 1+\frac{1}{2}\cdot\frac{9}{16}-\left(1+\frac{9}{16}\right)^{1/2} = \frac{1}{32}.$$

7b. Solve the differential equation $xy' = 2\sqrt{xy} - y$.

Solution. To turn this into a linear equation, make the substitution $u = \sqrt{xy}$. Then $xy = u^2$, and differentiating this gives $y + xy' = 2uu'$ so the equation becomes

$$2uu' = 2u.$$

One solution is given by $u = 0$, and the rest are given by $u' = 1$ (and hence $u = x + C$.) The first case gives the solution $y = 0$ and the second case gives the solution

$$y = \frac{(x+C)^2}{x}.$$