Sequences

Per-Olof Persson
persson@berkeley.edu

Department of Mathematics
University of California, Berkeley

Math 1B Calculus

Sequences

- A *sequence* is a list of numbers in a definite order:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

- The term a_n is called the *nth term*
- The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}$$

Limits

Definition

A sequence $\{a_n\}$ has the *limit* L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad |a_n - L| < \varepsilon$$

If $\lim_{n\to\infty} a_n$ exists, the sequence *converges* (or is *convergent*). Otherwise, the sequence *diverges* (or is *divergent*).

Definition

$\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad a_n > M$$

Limit Laws for Sequences

Theorem (Limit Laws for Sequences)

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$
$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \text{ if } \lim_{n\to\infty} b_n \neq 0$$
$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \text{ if } p > 0 \text{ and } a_n > 0$$

Useful Limit Theorems

Theorem

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

Theorem (Squeeze Theorem for Sequences)

If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Theorem

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Theorem

If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

Limits

Powers

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \qquad \text{if } r > 0$$

Exponentials

The sequence $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Increasing, Decreasing, and Bounded Sequences

Definition

A sequence $\{a_n\}$ is called *increasing* if $a_n < a_{n+1}$ for all $n \geq 1$, i.e., $a_1 < a_2 < a_3 < \cdots$. It is called *decreasing* if $a_n > a_{n+1}$ for all $n \geq 1$. It is called *monotonic* if it is either increasing or decreasing.

Definition

A sequence $\{a_n\}$

- is *bounded above* if there is a number M such that $a_n \leq M$ for all $n \geq 1$.
- is *bounded below* if there is a number m such that $m \leq a_n$ for all $n \geq 1$.
- is a *bounded sequence* if it is bounded above and below.

Theorem (Monotonic Sequence Theorem)

Every bounded, monotonic sequence is convergent.