

Sequences

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Math 1B Calculus

- A *sequence* is a list of numbers in a definite order:

$$a_1, a_2, a_3, \dots, a_n, \dots$$

- The term a_n is called the *n*th term
- The sequence $\{a_1, a_2, a_3, \dots\}$ is also denoted by

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

Definition

A sequence $\{a_n\}$ has the *limit* L and we write

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \text{ as } n \rightarrow \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if } n > N \quad \text{then} \quad |a_n - L| < \varepsilon$$

If $\lim_{n \rightarrow \infty} a_n$ exists, the sequence *converges* (or is *convergent*).
Otherwise, the sequence *diverges* (or is *divergent*).

Definition

$\lim_{n \rightarrow \infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if } n > N \quad \text{then} \quad a_n > M$$

Limit Laws for Sequences

Theorem (Limit Laws for Sequences)

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n \qquad \lim_{n \rightarrow \infty} c = c$$

$$\lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} \text{ if } \lim_{n \rightarrow \infty} b_n \neq 0$$

$$\lim_{n \rightarrow \infty} a_n^p = \left[\lim_{n \rightarrow \infty} a_n \right]^p \text{ if } p > 0 \text{ and } a_n > 0$$

Useful Limit Theorems

Theorem

If $\lim_{x \rightarrow \infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n \rightarrow \infty} a_n = L$.

Theorem (Squeeze Theorem for Sequences)

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$.

Theorem

If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Theorem

If $\lim_{n \rightarrow \infty} a_n = L$ and the function f is continuous at L , then

$$\lim_{n \rightarrow \infty} f(a_n) = f(L)$$

Powers

$$\lim_{n \rightarrow \infty} \frac{1}{n^r} = 0 \quad \text{if } r > 0$$

Exponentials

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r .

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Increasing, Decreasing, and Bounded Sequences

Definition

A sequence $\{a_n\}$ is called *increasing* if $a_n < a_{n+1}$ for all $n \geq 1$, i.e., $a_1 < a_2 < a_3 < \dots$. It is called *decreasing* if $a_n > a_{n+1}$ for all $n \geq 1$. It is called *monotonic* if it is either increasing or decreasing.

Definition

A sequence $\{a_n\}$

- is *bounded above* if there is a number M such that $a_n \leq M$ for all $n \geq 1$.
- is *bounded below* if there is a number m such that $m \leq a_n$ for all $n \geq 1$.
- is a *bounded sequence* if it is bounded above and below.

Theorem (Monotonic Sequence Theorem)

Every bounded, monotonic sequence is convergent.