17.4.3

$$y=\sum_{n=0}^{\infty} c_n x^n$$

$$y'=\sum_{n=1}^{\infty} n c_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n$$

$$x^2y=\sum_{n=0}^{\infty} c_n x^{n+2}=\sum_{n=2}^{\infty} c_{n-2}x^n$$

$$y'=x^2y \Rightarrow \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n=\sum_{n=2}^{\infty}c_{n-2}x^n$$

$n=0$: $1\cdot c_1=0 \Rightarrow c_1=0$

$n=1$: $2\cdot c_2=0 \Rightarrow c_2=0$

$n\geq 2$: $(n+1)c_{n+1}=c_{n-2} \Rightarrow c_{n+1}=\dfrac{c_{n-2}}{n+1}$

$$\Rightarrow c_1=c_4=c_7=\cdots=0$$

$$c_2=c_5=c_8=\cdots=0$$

$$c_3=\frac{c_0}{3},\quad c_6=\frac{c_3}{6}=\frac{c_0}{3\cdot 6},\quad c_9=\frac{c_6}{9}=\frac{c_0}{3\cdot 6\cdot 9},\ \ldots,\ c_{3n}=\frac{c_0}{3\cdot 6\cdots(3n)}=\frac{c_0}{3^n\cdot n!}$$

$$\Rightarrow y=\sum_{n=0}^{\infty}c_nx^n=c_0\sum_{n=0}^{\infty}\frac{x^{3n}}{3^n\cdot n!}$$

17.4.7

$$y=\sum_{n=0}^{\infty}c_nx^n$$

$$y'=\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n$$

$$y''=\sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}x^n$$

$$xy''=\sum_{n=1}^{\infty}n(n+1)c_{n+1}x^n$$

$(x-1)y''+y'=0 \Rightarrow$

$$\sum_{n=1}^{\infty}n(n+1)c_{n+1}x^n-\sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n=0$$

$n=0$: $-1\cdot 2\cdot c_2+1\cdot c_1=0 \Rightarrow c_2=\dfrac{c_1}{2}$

$n\geq 1$: $n(n+1)c_{n+1}-(n+1)(n+2)c_{n+2}+(n+1)c_{n+1}=0$

$$n\,c_{n+1}-(n+2)c_{n+2}+c_{n+1}=0 \Rightarrow c_{n+2}=\frac{n+1}{n+2}c_{n+1} \Rightarrow c_{n+1}=\frac{n}{n+1}c_n$$

$$\Rightarrow c_3=\frac{2}{3}c_2=\frac{c_1}{3}$$

$$c_4=\frac{3}{4}c_3=\frac{c_1}{4}$$

$$c_5=\frac{4}{5}c_4=\frac{c_1}{5}$$

$$\Rightarrow y=\sum_{n=0}^{\infty}c_nx^n=c_0+\sum_{n=1}^{\infty}\frac{c_1}{n}x^n=c_0+c_1\sum_{n=1}^{\infty}\frac{x^n}{n}$$

17.4.9

$$y=\sum_{n=0}^{\infty} c_n x^n, \quad y(0)=1 \Rightarrow c_0=1$$

$$y'=\sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y'(0)=0 \Rightarrow c_1=0$$

$$y''=\sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}x^n$$

$$xy'=\sum_{n=1}^{\infty} n c_n x^n$$

$$y''-xy'-y=0 \Rightarrow \sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}x^n - \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$n=0$: $1\cdot 2\cdot c_2 - c_0 = 0 \Rightarrow c_2 = \frac{c_0}{2}$

$n\geq 1$: $(n+1)(n+2)c_{n+2} - nc_n - c_n = 0 \Rightarrow c_{n+2} = \frac{c_n}{n+2}$

$\Rightarrow c_1 = c_3 = c_5 = \cdots = 0$

$\Rightarrow c_4 = \frac{c_2}{4} = \frac{c_0}{2\cdot 4}, \quad c_6 = \frac{c_4}{6} = \frac{c_0}{2\cdot 4\cdot 6}, \ldots, c_{2n} = \frac{c_0}{2\cdot 4\cdots(2n)} = \frac{1}{2^n\cdot n!}$

$$\Rightarrow y=\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty}\frac{x^{2n}}{2^n n!}$$

17.4.11

$$y=\sum_{n=0}^{\infty} c_n x^n, \quad xy=\sum_{n=0}^{\infty} c_n x^{n+1} = \sum_{n=1}^{\infty} c_{n-1}x^n \qquad y(0)=0 \Rightarrow c_0=0$$

$$y'=\sum_{n=1}^{\infty} n c_n x^{n-1}, \quad x^2y'=\sum_{n=1}^{\infty} n c_n x^{n+1} = \sum_{n=2}^{\infty}(n-1)c_{n-1}x^n \qquad y'(0)=1 \Rightarrow c_1=1$$

$$y''=\sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}x^n$$

$$y''+x^2y'+xy=0 \Rightarrow \sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}x^n + \sum_{n=2}^{\infty}(n-1)c_{n-1}x^n + \sum_{n=1}^{\infty}c_{n-1}x^n = 0$$

$n=0$: $1\cdot 2\cdot c_2 = 0 \Rightarrow c_2 = 0$

$n=1$: $2\cdot 3\cdot c_3 + c_0 = 0 \Rightarrow c_3 = 0$

$n\geq 2$: $(n+1)(n+2)c_{n+2} + (n-1)c_{n-1} + c_{n-1} = 0 \Rightarrow c_{n+2} = -\frac{n}{(n+1)(n+2)}c_{n-1}$

$\Rightarrow c_0 = c_3 = c_6 = \cdots = 0, \quad c_2 = c_5 = c_8 = \cdots = 0$

$c_1 = 1, \quad c_4 = -\frac{2}{3\cdot 4}c_1 = -\frac{2}{3\cdot 4}, \quad c_7 = -\frac{5}{6\cdot 7}c_4 = \frac{2\cdot 5}{3\cdot 4\cdot 6\cdot 7}, \quad c_{10} = \frac{-8}{9\cdot 10} = -\frac{2\cdot 5\cdot 8}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}$

$$\Rightarrow c_{3n+1} = (-1)^n \cdot \frac{2\cdot 5\cdot 8\cdots(3n-1)}{(3n+1)!/(2\cdot 5\cdot 8\cdots(3n-1))} = (-1)^n\cdot\frac{2^2\cdot 5^2\cdot 8^2\cdots(3n-1)^2}{(3n+1)!}$$

$$\Rightarrow y=\sum_{n=0}^{\infty}c_n x^n = x + \sum_{n=1}^{\infty}\frac{(-1)^n 2^2 5^2\cdots(3n-1)^2}{(3n+1)!}x^{3n+1}$$