1. For the two Diagonally Implicit Runge-Kutta (DIRK) schemes below:
   - Determine the order of the method
   - Find the stability function $R(z)$ and plot the stability region
   - Determine if the method is A-stable, L-stable, and algebraically stable/B-stable (but if a scheme is not algebraically stable just say that its B-stability is unclear)

Feel free to use MATLAB to check the conditions numerically, or Maple to do some of the algebra. Include brief but sufficient details in your write-up to show what you did.

\[
\begin{array}{ccc}
\alpha & \alpha & 0 \\
1 & 1-\alpha & \alpha \\
1-\alpha & \alpha & \\
\end{array}
\]

with $\alpha = 1 - \frac{\sqrt{2}}{2}$.

\[
\begin{array}{ccc}
\alpha & \alpha & 0 \\
1-\alpha & 1-2\alpha & \alpha \\
1/2 & 1/2 & \\
\end{array}
\]

with $\alpha = \frac{1}{2} + \frac{1}{2\sqrt{3}}$.

2. Prove that any implicit A-stable Runge-Kutta method with nonsingular $A$ satisfying $a_{sj} = b_j$ for all $j$ is L-stable.

3. Determine the order of the three-step method
   \[u_{n+3} - u_n = h \left[ \frac{3}{8}f_{n+3} + \frac{9}{8}f_{n+2} + \frac{9}{8}f_{n+1} + \frac{3}{8}f_n \right],\]
   the three-eights scheme. Is it convergent?

4. Show that the multistep method
   \[u_{n+3} + a_2u_{n+2} + a_1u_{n+1} + a_0u_n = h \left[ b_2f_{n+2} + b_1f_{n+1} + b_0f_n \right],\]
   is fourth order only if $a_0 + a_2 = 8$ and $a_1 = -9$. Hence, deduce that this method cannot be both fourth order and convergent.

5. A common argument is that BDF methods are more efficient than DIRK methods because they only require one solution of linear systems per timestep, while an $s$-stage DIRK method requires $s$ solutions. However, if a given DIRK method is more accurate per timestep than a corresponding BDF method, it could use a larger timestep for the same accuracy and it is not so clear which method is most efficient.

To investigate this effect for the linear test problem $y' = \lambda y$, compute the leading term of the one-step error $\tau(h)$ for the DIRK scheme in 1a and for BDF2. Use this to calculate the ratio $W_{\text{BDF2}}/W_{\text{DIRK}}$ between the work required for the two methods to obtain an error $\varepsilon$ at a time $T$. Assume the the work is dominated by the cost of solving the linear systems, and that the error $e(T, h) = C(T)\tau(h)/h$ with the same value of $C(T)$ for both methods. The answer should be a numerical value, independent of $\lambda$, $h$, $\varepsilon$, and $T$. 

6. In this problem, we will investigate the same effect we studied in 5, but numerically on a 2-D heat-transfer problem $u' = Ku$. The file heat2d_data.mat on the course web page contains the matrix $K$, the initial condition $u_0$, and the arrays $x_{in}, y_{in}$ which are only used for plotting. The function heat2d_plot.m can be used for plotting the solution $u$, the example below plots the initial condition:

```
load heat2d_data
heat2d_plot(xin,yin,u0);
```

This is useful for debugging your codes, but do not plot anything in your submitted codes to make them as fast as possible.

a) Implement a function

$$u = \text{heat2d\_rk4}(T,h)$$

which computes $u(T)$ using the explicit RK4 method. The timestep $h$ has to be small, since the method is explicit, but we will do this only once to get an “exact” solution. Test the function with the input values $T=2e-3$ and $h=1e-5$.

b) Implement a function

$$u = \text{heat2d\_dirk}(T,h)$$

which computes $u(T)$ using the DIRK scheme in 1a.

c) Implement a function

$$u = \text{heat2d\_bdf}(T,h)$$

which computes $u(T)$ using the BDF2 method. To get started with the multistep method, use backward-Euler for the first step.

d) Implement a function

$$[edirk,ebdf,wdirk,wbdf,wratio] = \text{heat2d\_conv}(T,hs)$$

which computes $u_{\text{exact}}$ at time $T$ using heat2d_rk4 with $h=1e-5$, and $udirk$, $ubdf$ using each timestep in $hs$. The outputs edirk, ebdf are the errors for each timestep $h$,

$$e_h(T) = \|u_h(T) - u_{\text{exact}}(T)\|_\infty$$

for the DIRK and the BDF2 methods, respectively. The outputs wdirk, wbdf are the number of linear solutions for the two methods, for each timestep. Finally, the wratio output is an estimate of the work ratio $W_{BDF}/W_{DIRK}$ for obtaining the same accuracy with the two methods (as in 5). Test the function using the commands below:

```
T=2e-3;
hs=1e-3*2.^(0:6);
[edirk,ebdf,wdirk,wbdf,wratio] = heat2d_conv(T,hs);
loglog(wdirk,edirk,wbdf,ebdf),grid on,legend('DIRK','BDF2')
xlabel('Work'),ylabel('Error')
wratio
```

Code Submission: E-mail the MATLAB files heat2d_rk4.m, heat2d_dirk.m, heat2d_bdf.m, heat2d_conv.m, and any supporting files to David at anderson@math.berkeley.edu as a zip-file named lastnameFirstname_4.zip, for example anderson_david_4.zip.