Chapter 2
Solutions of Equations in One Variable

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The Bisection Method

Suppose $f$ continuous on $[a, b]$, and $f(a)$, $f(b)$ opposite signs
By the IVT, there exists an $x$ in $(a, b)$ with $f(x) = 0$
Divide the interval $[a, b]$ by computing the midpoint

$$p = (a + b)/2$$

If $f(p)$ has same sign as $f(a)$, consider new interval $[p, b]$
If $f(p)$ has same sign as $f(b)$, consider new interval $[a, p]$
Repeat until interval small enough to approximate $x$ well
MATLAB Implementation

function p = bisection(f, a, b, tol)
% Solve f(p) = 0 using the bisection method.

while 1
    p = (a+b) / 2;
    if p−a < tol, break; end
    if f(a)*f(p) > 0
        a = p;
    else
        b = p;
    end
end
Termination Criteria

- Many ways to decide when to stop:

\[ |p_N - p_{N-1}| < \varepsilon \]
\[ \frac{|p_N - p_{N-1}|}{|p_N|} < \varepsilon \]
\[ |f(p_N)| < \varepsilon \]

- None is perfect, use a combination in real software
Theorem

Suppose that $f \in C[a, b]$ and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero $p$ of $f$ with

$$|p_n - p| \leq \frac{b - a}{2^n}, \quad \text{when } n \geq 1.$$ 

Convergence Rate

- The sequence $\{p_n\}_{n=1}^{\infty}$ converges to $p$ with rate of convergence $O(1/2^n)$:

$$p_n = p + O\left(\frac{1}{2^n}\right).$$
A number \( p \) is a \textit{fixed point} for a given function \( g \) if \( g(p) = p \).

Given a root-finding problem \( f(p) = 0 \), there are many \( g \) with fixed points at \( p \):

\[
g(x) = x - f(x) \\
g(x) = x + 3f(x) \\
\ldots
\]

If \( g \) has fixed point at \( p \), then \( f(x) = x - g(x) \) has a zero at \( p \).
Existence and Uniqueness of Fixed Points

Theorem

a. If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then $g$ has a fixed point in $[a, b]$

b. If, in addition, $g'(x)$ exists on $(a, b)$ and a positive constant $k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a, b),$$

then the fixed point in $[a, b]$ is unique.
Fixed-Point Iteration

For initial $p_0$, generate sequence $\{p_n\}_{n=0}^{\infty}$ by $p_n = g(p_{n-1})$. If the sequence converges to $p$, then

$$p = \lim_{n \to \infty} p_n = \lim_{n \to \infty} g(p_{n-1}) = g \left( \lim_{n \to \infty} p_{n-1} \right) = g(p).$$

MATLAB Implementation

```matlab
function p = fixedpoint(g, p0, tol)
% Solve g(p) = p using fixed-point iteration.

while 1
    p = g(p0);
    if abs(p-p0) < tol, break; end
    p0 = p;
end
```
Convergence of Fixed-Point Iteration

**Theorem (Fixed-Point Theorem)**

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all $x$ in $[a, b]$. Suppose, in addition, that $g'$ exists on $(a, b)$ and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a, b).$$

Then, for any number $p_0$ in $[a, b]$, the sequence defined by $p_n = g(p_{n-1})$ converges to the unique fixed point $p$ in $[a, b]$.

**Corollary**

If $g$ satisfies the hypotheses above, then bounds for the error are given by

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$

$$|p_n - p| \leq \frac{k^n}{1 - k}|p_1 - p_0|$$
Newton’s Method

Taylor Polynomial Derivation

Suppose $f \in C^2[a, b]$ and $p_0 \in [a, b]$ approximates solution $p$ of $f(x) = 0$ with $f'(p_0) \neq 0$. Expand $f(x)$ about $p_0$:

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p))$$

Set $f(p) = 0$, assume $(p - p_0)^2$ negligible:

$$p \approx p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}$$

This gives the sequence $\{p_n\}_{n=0}^{\infty}$:

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$
function p = newton(f, df, p0, tol)
% Solve f(p) = 0 using Newton's method.

while 1
    p = p0 - f(p0)/df(p0);
    if abs(p-p0) < tol, break; end
    p0 = p;
end
Newton’s Method – Convergence

Fixed Point Formulation

Newton’s method is fixed point iteration $p_n = g(p_{n-1})$ with

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Theorem

Let $f \in C^2[a, b]$. If $p \in [a, b]$ is such that $f(p) = 0$ and $f'(p) \neq 0$, then there exists a $\delta > 0$ such that Newton’s method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to $p$ for any initial approximation $p_0 \in [p - \delta, p + \delta]$. 
The Secant Method

Replace the derivative in Newton’s method by

\[ f'(p_{n-1}) \approx \frac{f(p_{n-2}) - f(p_{n-1})}{p_{n-2} - p_{n-1}} \]

to get

\[ p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})} \]

The Method of False Position (Regula Falsi)

Like the Secant method, but with a test to ensure the root is bracketed between iterations.
Order of Convergence

Definition

Suppose \( \{p_n\}_{n=0}^{\infty} \) is a sequence that converges to \( p \), with \( p_n \neq p \) for all \( n \). If positive constants \( \lambda \) and \( \alpha \) exist with

\[
\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda,
\]

then \( \{p_n\}_{n=0}^{\infty} \) converges to \( p \) of order \( \alpha \), with asymptotic error constant \( \lambda \).

An iterative technique \( p_n = g(p_{n-1}) \) is said to be of order \( \alpha \) if the sequence \( \{p_n\}_{n=0}^{\infty} \) converges to the solution \( p = g(p) \) of order \( \alpha \).

Special cases

- If \( \alpha = 1 \) (and \( \lambda < 1 \)), the sequence is *linearly convergent*
- If \( \alpha = 2 \), the sequence is *quadratically convergent*
**Theorem**

Let \( g \in C[a, b] \) be such that \( g(x) \in [a, b] \), for all \( x \in [a, b] \). Suppose \( g' \) is continuous on \((a, b)\) and that \( 0 < k < 1 \) exists with \( |g'(x)| \leq k \) for all \( x \in (a, b) \). If \( g'(p) \neq 0 \), then for any number \( p_0 \) in \([a, b]\), the sequence \( p_n = g(p_{n-1}) \) converges only linearly to the unique fixed point \( p \) in \([a, b]\).

**Theorem**

Let \( p \) be solution of \( x = g(x) \). Suppose \( g'(p) = 0 \) and \( g'' \) continuous with \( |g''(x)| < M \) on open interval \( I \) containing \( p \). Then there exists \( \delta > 0 \) s.t. for \( p_0 \in [p - \delta, p + \delta] \), the sequence defined by \( p_n = g(p_{n-1}) \) converges at least quadratically to \( p \), and

\[
|p_{n+1} - p| < \frac{M}{2} |p_n - p|^2.
\]
Newton’s Method as Fixed-Point Problem

**Derivation**

Seek $g$ of the form

$$g(x) = x - \phi(x)f(x).$$

Find differentiable $\phi$ giving $g'(p) = 0$ when $f(p) = 0$:

$$g'(x) = 1 - \phi'(x)f(x) - f'(x)\phi(x)$$

$$g'(p) = 1 - \phi'(p) \cdot 0 - f'(p)\phi(p)$$

and $g'(p) = 0$ if and only if $\phi(p) = 1/f'(p)$. This gives Newton’s method

$$p_n = g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$
Definition
A solution $p$ of $f(x) = 0$ is a zero of multiplicity $m$ of $f$ if for $x \neq p$, we can write $f(x) = (x - p)^m q(x)$, where $\lim_{x \to p} q(x) \neq 0$.

Theorem
$f \in C^1[a, b]$ has a simple zero at $p$ in $(a, b)$ if and only if $f(p) = 0$, but $f'(p) \neq 0$.

Theorem
The function $f \in C^m[a, b]$ has a zero of multiplicity $m$ at point $p$ in $(a, b)$ if and only if

$$0 = f(p) = f'(p) = f''(p) = \cdots = f^{(m-1)}(p), \text{ but } f^{(m)}(p) \neq 0.$$
Newton’s Method for Multiple Roots

Define $\mu(x) = \frac{f(x)}{f'(x)}$. If $p$ is a zero of $f$ of multiplicity $m$ and $f(x) = (x - p)^mq(x)$, then

$$\mu(x) = (x - p)\frac{q(x)}{mq(x) + (x - p)q'(x)}$$

also has a zero at $p$. But $q(p) \neq 0$, so

$$\frac{q(p)}{mq(p) + (p - p)q'(p)} = \frac{1}{m} \neq 0,$$

and $p$ is a simple zero of $\mu$. Newton’s method can then be applied to $\mu$ to give

$$g(x) = x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)}$$
Aitken’s $\Delta^2$ Method

Accelerating linearly convergent sequences

- Suppose $\{p_n\}_{n=0}^{\infty}$ linearly convergent with limit $p$
- Assume that
  \[
  \frac{p_{n+1} - p}{p_n - p} \approx \frac{p_{n+2} - p}{p_{n+1} - p}
  \]
- Solving for $p$ gives
  \[
  p \approx \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n} = \cdots = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}
  \]
- Use this for new more rapidly converging sequence $\{\hat{p}_n\}_{n=0}^{\infty}$:
  \[
  \hat{p}_n = p_n - \frac{(p_{n+1} - p)^2}{p_{n+2} - 2p_{n+1} + p_n}
  \]
Delta Notation

**Definition**

For a given sequence \( \{p_n\}_{n=0}^{\infty} \), the *forward difference* \( \Delta p_n \) is defined by

\[
\Delta p_n = p_{n+1} - p_n, \quad \text{for } n \geq 0
\]

Higher powers of the operator \( \Delta \) are defined recursively by

\[
\Delta^k p_n = \Delta(\Delta^{k-1} p_n), \quad \text{for } k \geq 2
\]

**Aitken’s \( \Delta^2 \) method using delta notation**

Since \( \Delta^2 p_n = p_{n+2} - 2p_{n+1} + p_n \), we can write

\[
\hat{p}_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n}, \quad \text{for } n \geq 0
\]
Convergence of Aitken’s $\Delta^2$ Method

**Theorem**

Suppose that $\{p_n\}_{n=0}^\infty$ converges linearly to $p$ and that

$$\lim_{n\to\infty} \frac{p_{n+1} - p}{p_n - p} < 1$$

Then $\{\hat{p}_n\}_{n=0}^\infty$ converges to $p$ faster than $\{p_n\}_{n=0}^\infty$ in the sense that

$$\lim_{n\to\infty} \frac{\hat{p}_n - p}{p_n - p} = 0$$
Steffensen’s Method

Accelerating fixed-point iteration

Aitken’s $\Delta^2$ method for fixed-point iteration gives

$$p_0, \ p_1 = g(p_0), \ p_2 = g(p_1), \ \hat{p}_0 = \{\Delta^2\}(p_0),$$
$$p_3 = g(p_2), \ \hat{p}_1 = \{\Delta^2\}(p_1), \ldots$$

Steffensen’s method assumes $\hat{p}_0$ is better than $p_2$:

$$p_0^{(0)}, \ p_1^{(0)} = g(p_0^{(0)}), \ p_2^{(0)} = g(p_1^{(0)}), \ p_0^{(1)} = \{\Delta^2\}(p_0^{(0)}),$$
$$p_1^{(1)} = g(p_0^{(1)}), \ldots$$

Theorem

Suppose $x = g(x)$ has solution $p$ with $g'(p) \neq 1$. If exists $\delta > 0$ s.t. $g \in C^3[p - \delta, p + \delta]$, then Steffensen’s method gives quadratic convergence for $p_0 \in [p - \delta, p + \delta]$. 
function p = steffensen(g, p0, tol)
% Solve g(p) = p using Steffensen's method.

while 1
    p1 = g(p0);
    p2 = g(p1);
    p = p0 - (p1-p0)^2 / (p2-2*p1+p0);
    if abs(p-p0) < tol, break; end
    p0 = p;
end