Lipschitz Condition and Convexity

**Definition**

A function $f(t, y)$ is said to satisfy a *Lipschitz condition* in the variable $y$ on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists with

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|,$$

whenever $(t, y_1), (t, y_2) \in D$. The constant $L$ is called a *Lipschitz constant* for $f$.

**Definition**

A set $D \subset \mathbb{R}^2$ is said to be *convex* if whenever $(t_1, y_1)$ and $(t_2, y_2)$ belong to $D$ and $\lambda$ is in $[0, 1]$, the point

$$((1 - \lambda)t_1 + \lambda t_2, (1 - \lambda)y_1 + \lambda y_2)$$

also belongs to $D$. 
Existence and Uniqueness

**Theorem**

Suppose \( f(t, y) \) is defined on a convex set \( D \subset \mathbb{R}^2 \). If a constant \( L > 0 \) exists with

\[
\left| \frac{\partial f}{\partial y} (t, y) \right| \leq L, \text{ for all } (t, y) \in D,
\]

then \( f \) satisfies a Lipschitz condition on \( D \) in the variable \( y \) with Lipschitz constant \( L \).

**Theorem**

Suppose that \( D = \{ (t, y) \mid a \leq t \leq b, -\infty < y < \infty \} \) and that \( f(t, y) \) is continuous on \( D \). If \( f \) satisfies a Lipschitz condition on \( D \) in the variable \( y \), then the initial-value problem

\[
y'(t) = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,
\]

has a unique solution \( y(t) \) for \( a \leq t \leq b \).
Well-Posedness

**Definition**

The initial-value problem

\[
\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,
\]

is said to be a well-posed problem if:

- A unique solution, \( y(t) \), to the problem exists, and
- There exist constants \( \varepsilon_0 > 0 \) and \( k > 0 \) such that for any \( \varepsilon \), with \( \varepsilon_0 > \varepsilon > 0 \), whenever \( \delta(t) \) is continuous with \( |\delta(t)| < \varepsilon \) for all \( t \) in \([a, b]\), and when \( |\delta_0| < \varepsilon \), the initial-value problem

\[
\frac{dz}{dt} = f(t, z) + \delta(t), \quad a \leq t \leq b, \quad z(a) = \alpha + \delta_0,
\]

has a unique solution \( z(t) \) that satisfies

\[
|z(t) - y(t)| < k\varepsilon \quad \text{for all } t \text{ in } [a, b].
\]
Well-Posedness

**Theorem**

Suppose \( D = \{(t, y) \mid a \leq t \leq b \text{ and } -\infty < y < \infty\} \). If \( f \) is continuous and satisfies a Lipschitz condition in the variable \( y \) on the set \( D \), then the initial-value problem

\[
\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha
\]

is well-posed.
Suppose a well-posed initial-value problem is given:

\[ \frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha \]

Distribute mesh points equally throughout \([a, b] \):

\[ t_i = a + ih, \quad \text{for each } i = 0, 1, 2, \ldots, N. \]

The step size \( h = (b - a)/N = t_{i+1} - t_i \).
Use Taylor’s Theorem for $y(t)$:

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2}y''(\xi_i)$$

for $\xi_i \in (t_i, t_{i+1})$. Since $h = t_{i+1} - t_i$ and $y'(t_i) = f(t_i, y(t_i))$,

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}y''(\xi_i).$$

Neglecting the remainder term gives Euler’s method for $w_i \approx y(t_i)$:

$$w_0 = \alpha$$

$$w_{i+1} = w_i + hf(t_i, w_i), \quad i = 0, 1, \ldots, N - 1$$

The well-posedness implies that

$$f(t_i, w_i) \approx y'(t_i) = f(t_i, y(t_i))$$
Suppose $f$ is continuous and satisfies a Lipschitz condition with constant $L$ on

$$D = \{(t, y) \mid a \leq t \leq b, -\infty < y < \infty\}$$

and that a constant $M$ exists with

$$|y''(t)| \leq M, \quad \text{for all } t \in [a, b].$$

Let $y(t)$ denote the unique solution to the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

and $w_0, w_1, \ldots, w_n$ as in Euler's method. Then

$$|y(t_i) - w_i| \leq \frac{hM}{2L} \left[ e^{L(t_i-a)} - 1 \right].$$
Local Truncation Error

**Definition**

The difference method

\[
\begin{align*}
   w_0 &= \alpha \\
   w_{i+1} &= w_i + h\phi(t_i, w_i)
\end{align*}
\]

has local truncation error

\[
\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\phi(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i),
\]

for each \( i = 0, 1, \ldots, N - 1. \)
Consider initial-value problem

\[ y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha. \]

Expand \( y(t) \) in \( n \)th Taylor polynomial about \( t_i \), evaluated at \( t_{i+1} \):

\[
y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(t_i) + \cdots \\
+ \frac{h^n}{n!} y^{(n)}(t_i) + \frac{h^{n+1}}{(n + 1)!} y^{(n+1)}(\xi_i) \\
= y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(t_i, y(t_i)) + \cdots \\
= \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n + 1)!} f^{(n)}(\xi_i, y(\xi_i))
\]

for some \( \xi_i \in (t_i, t_{i+1}) \). Delete remainder term to obtain the Taylor method of order \( n \).
Taylor Method of Order $n$

\[ w_0 = \alpha \]
\[ w_{i+1} = w_i + hT^{(n)}(t_i, w_i), \quad i = 0, 1, \ldots, N - 1 \]

where

\[ T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{(n-1)}}{n!} f^{(n-1)}(t_i, w_i) \]
Theorem

If Taylor’s method of order $n$ is used to approximate the solution to

$$y'(t) = f(t, y(t)), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

with step size $h$ and if $y \in C^{n+1}[a, b]$, then the local truncation error is $O(h^n)$. 
Taylor’s Theorem in Two Variables

**Theorem**

Suppose $f(t, y)$ and partial derivatives up to order $n + 1$ continuous on $D = \{ (t, y) \mid a \leq t \leq b, c \leq y \leq d \}$, let $(t_0, y_0) \in D$. For $(t, y) \in D$, there is $\xi \in [t, t_0]$ and $\mu \in [y, y_0]$ with

$$f(t, y) = P_n(t, y) + R_n(t, y)$$

$$P_n(t, y) = f(t_0, y_0) + \left[ (t - t_0) \frac{\partial f}{\partial t}(t_0, y_0) + (y - y_0) \frac{\partial f}{\partial y}(t_0, y_0) \right]$$

$$+ \left[ \frac{(t - t_0)^2}{2} \frac{\partial^2 f}{\partial t^2}(t_0, y_0) + (t - t_0)(y - y_0) \frac{\partial^2 f}{\partial t \partial y}(t_0, y_0) \right]$$

$$+ \frac{(y - y_0)^2}{2} \frac{\partial^2 f}{\partial y^2}(t_0, y_0) \right] + \cdots$$

$$+ \left[ \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} (t - t_0)^{n-j}(y - y_0)^j \frac{\partial^n f}{\partial t^{n-j} \partial y^j}(t_0, y_0) \right]$$
Taylor’s Theorem in Two Variables

Theorem

(cont’d)

\[ R_n(t, y) = \frac{1}{(n + 1)!} \sum_{j=0}^{n+1} \binom{n + 1}{j} (t - t_0)^{n+1-j} (y - y_0)^j \cdot \frac{\partial^{n+1} f}{\partial t^{n+1-j} \partial y^j} (\xi, \mu) \]

\[ P_n(t, y) \text{ is the } n\text{th Taylor polynomial in two variables.} \]
Obtain high-order accuracy of Taylor methods without knowledges of derivatives of $f$.

Determine $a_1, \alpha_1, \beta_1$ such that

$$a_1 f(t + \alpha_1, y + \beta_1) \approx f(t, y) + \frac{h}{2} f'(t, y) = T^{(2)}(t, y).$$

with $O(h^2)$ error.

Since

$$f'(t, y) = \frac{df}{dt}(t, y) = \frac{\partial f}{\partial t}(t, y) + \frac{\partial f}{\partial y}(t, y) \cdot y'(t)$$

and $y'(t) = f(t, y)$, we have

$$T^{(2)}(t, y) = f(t, y) + \frac{h}{2} \frac{\partial f}{\partial t}(t, y) + \frac{h}{2} \frac{\partial f}{\partial y}(t, y) \cdot f(t, y)$$
Expand \( f(t + \alpha_1, y + \beta_1) \) in 1st degree Taylor polynomial:

\[
a_1 f(t + \alpha_1, y + \beta_1) = a_1 f(t, y) + a_1 \alpha_1 \frac{\partial f}{\partial t}(t, y) \\
+ a_1 \beta_1 \frac{\partial f}{\partial y}(t, y) + a_1 \cdot R_1(t + \alpha_1, y + \beta_1)
\]

Matching coefficients gives

\[
a_1 = 1 \quad a_1 \alpha_1 = \frac{h}{2}, \quad a_1 \beta_1 = \frac{h}{2} f(t, y)
\]

with unique solution

\[
a_1 = 1, \quad \alpha_1 = \frac{h}{2}, \quad \beta_1 = \frac{h}{2} f(t, y)
\]
Runge-Kutta Methods

- This gives

\[ T^{(2)}(t, y) = f \left( t + \frac{h}{2}, y + \frac{h}{2} f(t, y) \right) - R_1 \left( t + \frac{h}{2}, y + \frac{h}{2} f(t, y) \right) \]

with \( R_1(\cdot, \cdot) = O(h^2) \)

Midpoint Method

- \( w_0 = \alpha, \)
- \( w_{i+1} = w_i + h f \left( t + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i) \right), \quad i = 0, 1, \ldots, N - 1 \)

Local truncation error of order two.
Runge-Kutta Methods

Runge-Kutta Order Four

\[
\begin{align*}
  w_0 &= \alpha \\
  k_1 &= hf(t_i, w_i) \\
  k_2 &= hf \left( t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1 \right) \\
  k_3 &= hf \left( t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2 \right) \\
  k_4 &= hf(t_{i+1}, w_i + k_3) \\
  w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{align*}
\]

Local truncation error \( O(h^4) \)
function [t, w] = rk4(f, a, b, alpha, N)
% Solve ODE y'(t) = f(t, y(t)) using Runge–Kutta 4.

h = (b−a) / N;
t = (a:h:b)';
w = zeros(N+1, length(alpha));
w(1,:) = alpha(:)';
for i = 1:N
    k1 = h*f(t(i), w(i,:));
k2 = h*f(t(i) + h/2, w(i,:) + k1/2);
k3 = h*f(t(i) + h/2, w(i,:) + k2/2);
k4 = h*f(t(i) + h, w(i,:) + k3);
w(i+1,:) = w(i,:) + (k1 + 2*k2 + 2*k3 + k4)/6;
end
**Multistep Methods**

**Definition**

An *m*-step multistep method for solving the initial-value problem

\[ y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha, \]

has a difference equation for approximate \( w_{i+1} \) at \( t_{i+1} \):

\[
w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0w_{i+1-m} \\
+ h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots \\
+ b_0 f(t_{i+1-m}, w_{i+1-m})],
\]

where \( h = (b - a)/N \), and starting values are specified:

\[
w_0 = \alpha, \quad w_1 = \alpha_1, \quad \ldots, \quad w_{m-1} = \alpha_{m-1}
\]

*Explicit* method if \( b_m = 0 \), *implicit* method if \( b_m \neq 0 \).
Multistep Methods

**Fourth-Order Adams-Bashforth Technique**

\[ w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3, \]
\[ w_{i+1} = w_i + \frac{h}{24} [55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) \]
\[ + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})] \]

**Fourth-Order Adams-Moulton Technique**

\[ w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \]
\[ w_{i+1} = w_i + \frac{h}{24} [9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) \]
\[ - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})] \]
Integrate the initial-value problem

\[ y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha \]

over \([t_i, t_{i+1}]:\)

\[ y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t)) \, dt \]

Replace \(f\) by polynomial \(P(t)\) interpolating \((t_0, w_0), \ldots, (t_i, w_i)\), and approximate \(y(t_i) \approx w_i: \)

\[ y(t_{i+1}) \approx w_i + \int_{t_i}^{t_{i+1}} P(t) \, dt \]
Adams-Bashforth explicit \( m \)-step: Newton backward-difference polynomial through
\((t_i, f(t_i, y(t_i))), \ldots, (t_{i+1-m}, f(t_{i+1-m}, y(t_{i+1-m})))\).

\[
\int_{t_i}^{t_{i+1}} f(t, y(t)) \, dt \approx \int_{t_i}^{t_{i+1}} \sum_{k=0}^{m-1} (-1)^k \binom{-s}{k} \nabla^k f(t_i, y(t_i)) \, dt
\]

\[
= \sum_{k=0}^{m-1} \nabla^k f(t_i, y(t_i)) h (-1)^k \int_0^1 \binom{-s}{k} \, ds
\]

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
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<td>((-1)^k \int_0^1 \binom{-s}{k} , ds)</td>
<td>1</td>
<td>(\frac{1}{2})</td>
<td>(\frac{5}{12})</td>
<td>(\frac{3}{8})</td>
<td>(\frac{251}{720})</td>
<td>(\frac{95}{288})</td>
</tr>
</tbody>
</table>
Three-step Adams-Bashforth:

\[ y(t_{i+1}) \approx y(t_i) + h \left[ f(t_i, y(t_i)) + \frac{1}{2} \nabla f(t_i, y(t_i)) + \frac{5}{12} \nabla^2 f(t_i, y(t_i)) \right] \]

\[ = y(t_i) + \frac{h}{12} \left[ 23f(t_i, y(t_i)) - 16f(t_{i-1}, y(t_{i-1})) + 5f(t_{i-2}, y(t_{i-2})) \right] \]

The method is:

\[ w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \]

\[ w_{i+1} = w_i + \frac{h}{12} \left[ 23f(t_i, w_i) - 16f(t_{i-1}, w_{i-1}) + 5f(t_{i-2}, w_{i-2}) \right] \]
Local Truncation Error

**Definition**

If \( y(t) \) solves

\[
y'(t, y) = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,
\]

and

\[
w_{i+1} = a_{m-1}w_i + \cdots + a_0 w_{i+1-m} \\
+ h[b_m f(t_{i+1}, w_{i+1}) + \cdots + b_0 f(t_{i+1-m}, w_{i+1-m})],
\]

the **local truncation error** is

\[
\tau_{i+1}(h) = \frac{y(t_{i+1}) - a_{m-1}y(t_i) - \cdots - a_0y(t_{i+1-m})}{h} \\
- [b_m f(t_{i+1}, y(t_{i+1})) + \cdots + b_0 f(t_{i+1-m}, y(t_{i+1-m}))].
\]
An $m$th-order system of first-order initial-value problems has the form

$$\frac{du_1}{dt}(t) = f_1(t, u_1, u_2, \ldots, u_m),$$

$$\frac{du_2}{dt}(t) = f_2(t, u_1, u_2, \ldots, u_m),$$

$$\vdots$$

$$\frac{du_m}{dt}(t) = f_m(t, u_1, u_2, \ldots, u_m),$$

for $a \leq t \leq b$, with the initial conditions

$$u_1(a) = \alpha_1, u_2(a) = \alpha_2, \ldots, u_m(a) = \alpha_m.$$
Existence and Uniqueness

Definition

The function \( f(t, y_1, \ldots, y_m) \), defined on the set

\[ D = \{ (t, u_1, \ldots, u_m) \mid a \leq t \leq b, -\infty < u_i < \infty, i = 1, 2, \ldots, m \} \]

is said to satisfy a \textit{Lipschitz condition} on \( D \) in the variables \( u_1, u_2, \ldots, u_m \) if a constant \( L > 0 \) exists with

\[
|f(t, u_1, \ldots, u_m) - f(t, z_1, \ldots, z_m)| \leq L \sum_{j=1}^{m} |u_j - z_j|,
\]

for all \((t, u_1, \ldots, u_m)\) and \((t, z_1, \ldots, z_m)\) in \( D \).
Existence and Uniqueness

Theorem

Suppose

\[ D = \{(t, u_1, \ldots, u_m) \mid a \leq t \leq b, -\infty < u_i < \infty, i = 1, 2, \ldots, m\} \]

and let \( f_i(t, u_1, \ldots, u_m) \), for each \( i = 1, 2, \ldots, m \), be continuous on \( D \) and satisfy a Lipschitz condition there. The system of first-order differential equations

\[ \frac{du_k}{dt}(t) = f_k(t, u_1, \ldots, u_m), \quad u_k(a) = \alpha_k, \quad k = 1, \ldots, m \]

has a unique solution \( u_1(t), \ldots, u_m(t) \) for \( a \leq t \leq b \).

Fourth order Runge-Kutta for systems

\[ w_0 = \alpha \]
\[ k_1 = hf(t_i, w_i) \]
\[ k_2 = hf(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1) \]
\[ k_3 = hf(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2) \]
\[ k_4 = hf(t_{i+1}, w_i + k_3) \]
\[ w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \]

where \( w_i = (w_{i,1}, \ldots, w_{i,m}) \) is the vector of unknowns.
**Definition**

A one-step difference-equation with local truncation error $\tau_i(h)$ is said to be *consistent* if

$$\lim_{h \to 0} \max_{1 \leq i \leq N} |\tau_i(h)| = 0$$

**Definition**

A one-step difference equation is said to be *convergent* if

$$\lim_{h \to 0} \max_{1 \leq i \leq N} |w_i - y(t_i)| = 0,$$

where $y_i = y(t_i)$ is the exact solution and $w_i$ the approximation.
Convergence of One-Step Methods

Theorem

Suppose the initial-value problem $y' = f(t, y), a \leq t \leq b$, $y(a) = \alpha$ is approximated by a one-step difference method in the form $w_0 = \alpha, w_{i+1} = w_i + h\phi(t_i, w_i, h)$. Suppose also that $h_0 > 0$ exists and $\phi(t, w, h)$ is continuous with a Lipschitz condition in $w$ with constant $L$ on $D$, then

$$D = \{(t, w, h) \mid a \leq t \leq b, -\infty < w < \infty, 0 \leq h \leq h_0\}.$$

1. The method is stable;
2. The method is convergent if and only if it is consistent:

$$\phi(t, y, 0) = f(t, y)$$

3. If $\tau$ exists s.t. $|\tau_i(h)| \leq \tau(h)$ when $0 \leq h \leq h_0$, then

$$|y(t_i) - w_i| \leq \frac{\tau(h)}{L} e^{L(t_i-a)}.$$
Let \( \lambda_1, \ldots, \lambda_m \) denote the roots of the characteristic equation

\[
P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - \cdots - a_1\lambda - a_0 = 0
\]

associated with the multistep method

\[
w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0w_{i+1-m} + hF(t_i, h, w_{i+1}, w_i, \ldots, w_{i+1-m}).
\]

If \( |\lambda_i| \leq 1 \) and all roots with absolute value 1 are simple, the method is said to satisfy the root condition.
Stability

**Definition**

1. Methods that satisfy the root condition and have \( \lambda = 1 \) as the only root of magnitude one are called **strongly stable**.
2. Methods that satisfy the root condition and have more than one distinct root with magnitude one are called **weakly stable**.
3. Methods that do not satisfy the root condition are **unstable**.

**Theorem**

A *multistep method*

\[
    w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \cdots + a_0 w_{i+1-m} \\
    + hF(t_i, h, w_{i+1}, w_i, \ldots, w_{i+1-m})
\]

is stable if and only if it satisfies the root condition. If it is also consistent, then it is stable if and only if it is convergent.
A **stiff differential equation** is numerically unstable unless the step size is extremely small.

Large derivatives give error terms that are dominating the solution.

Example: The initial-value problem

\[ y' = -30y, \quad 0 \leq t \leq 1.5, \quad y(0) = \frac{1}{3} \]

has exact solution \( y = \frac{1}{3} e^{-30t} \). But RK4 is unstable with step size \( h = 0.1 \).
Euler’s Method for Test Equation

- Consider the simple test equation

\[ y' = \lambda y, \quad y(0) = \alpha, \quad \text{where } \lambda < 0 \]

with solution \( y(t) = \alpha e^{\lambda t} \).

- Euler’s method gives \( w_0 = \alpha \) and

\[ w_{j+1} = w_j + h(\lambda w_j) = (1 + h\lambda)w_j = (1 + h\lambda)^{j+1}\alpha. \]

- The absolute error is

\[ |y(t_j) - w_j| = \left| e^{jh\lambda} - (1 + h\lambda)^j \right| |\alpha| \]

\[ = \left| (e^{h\lambda})^j - (1 + h\lambda)^j \right| |\alpha| \]

- Stability requires \(|1 + h\lambda| < 1\), or \(h < 2/|\lambda|\).
Apply a multistep method to the test equation:

\[ w_{j+1} = a_{m-1}w_j + \cdots + a_0w_{j+1-m} \\
+ h\lambda (b_mw_{j+1} + b_{m-1}w_j + \cdots + b_0w_{j+1-m}) \]

or

\[ (1 - h\lambda b_m)w_{j+1} - (a_{m-1} + h\lambda b_{m-1})w_j - \cdots - (a_0 + h\lambda b_0)w_{j+1-m} = 0 \]

Let \( \beta_1, \ldots, \beta_m \) be the zeros of the characteristic polynomial

\[ Q(z, h\lambda) = (1 - h\lambda b_m)z^m - (a_{m-1} + h\lambda b_{m-1})z^{m-1} - \cdots - (a_0 + h\lambda b_0) \]

Then \( c_1, \ldots, c_m \) exist with

\[ w_j = \sum_{k=1}^{m} c_k (\beta_k)^j \]

and \( |\beta_k| < 1 \) is required for stability.
The region $R$ of absolute stability for a one-step method is

$$R = \{ h\lambda \in \mathbb{C} \mid |Q(h\lambda)| < 1 \},$$

and for a multistep method, it is

$$R = \{ h\lambda \in \mathbb{C} \mid |\beta_k| < 1, \text{ for all zeros } \beta_k \text{ of } Q(z, h\lambda) \}.$$