Chapter 5 – Initial-Value Problems for Ordinary Differential Equations

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Definition

A function $f(t, y)$ is said to satisfy a **Lipschitz condition** in the variable $y$ on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists with

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|,$$

whenever $(t, y_1), (t, y_2) \in D$. The constant $L$ is called a **Lipschitz constant** for $f$.

Definition

A set $D \subset \mathbb{R}^2$ is said to be **convex** if whenever $(t_1, y_1)$ and $(t_2, y_2)$ belong to $D$ and $\lambda$ is in $[0, 1]$, the point

$$((1 - \lambda)t_1 + \lambda t_2, (1 - \lambda)y_1 + \lambda y_2)$$

also belongs to $D$. 
Existence and Uniqueness

**Theorem**

Suppose \( f(t, y) \) is defined on a convex set \( D \subset \mathbb{R}^2 \). If a constant \( L > 0 \) exists with

\[
\left| \frac{\partial f}{\partial y}(t, y) \right| \leq L, \quad \text{for all} \ (t, y) \in D,
\]

then \( f \) satisfies a Lipschitz condition on \( D \) in the variable \( y \) with Lipschitz constant \( L \).

**Theorem**

Suppose that \( D = \{(t, y) \mid a \leq t \leq b, -\infty < y < \infty \} \) and that \( f(t, y) \) is continuous on \( D \). If \( f \) satisfies a Lipschitz condition on \( D \) in the variable \( y \), then the initial-value problem

\[
y'(t) = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,
\]

has a unique solution \( y(t) \) for \( a \leq t \leq b \).
The initial-value problem
\[
\frac{dy}{dt} = f(t,y), \quad a \leq t \leq b, \quad y(a) = \alpha,
\]
is said to be a well-posed problem if:
- A unique solution, \(y(t)\), to the problem exists, and
- There exist constants \(\varepsilon_0 > 0\) and \(k > 0\) such that for any \(\varepsilon\), with \(\varepsilon_0 > \varepsilon > 0\), whenever \(\delta(t)\) is continuous with \(|\delta(t)| < \varepsilon\) for all \(t\) in \([a, b]\), and when \(|\delta_0| < \varepsilon\), the initial-value problem
\[
\frac{dz}{dt} = f(t,z) + \delta(t), \quad a \leq t \leq b, \quad z(a) = \alpha + \delta_0,
\]
has a unique solution \(z(t)\) that satisfies
\[
|z(t) - y(t)| < k\varepsilon \quad \text{for all } t \text{ in } [a, b].
\]
Well-Posedness

**Theorem**

Suppose \( D = \{(t, y) \mid a \leq t \leq b \text{ and } -\infty < y < \infty\} \). If \( f \) is continuous and satisfies a Lipschitz condition in the variable \( y \) on the set \( D \), then the initial-value problem

\[
\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha
\]

is well-posed.
Suppose a well-posed initial-value problem is given:

\[ \frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha \]

Distribute mesh points equally throughout \([a, b]\):

\[ t_i = a + ih, \quad \text{for each } i = 0, 1, 2, \ldots, N. \]

The *step size* \( h = (b - a)/N = t_{i+1} - t_i. \)
Use Taylor’s Theorem for $y(t)$:

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2} y''(\xi_i)$$

for $\xi_i \in (t_i, t_{i+1})$. Since $h = t_{i+1} - t_i$ and $y'(t_i) = f(t_i, y(t_i))$,

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i).$$

Neglecting the remainder term gives Euler’s method for $w_i \approx y(t_i)$:

$$w_0 = \alpha$$

$$w_{i+1} = w_i + hf(t_i, w_i), \quad i = 0, 1, \ldots, N - 1$$

The well-posedness implies that

$$f(t_i, w_i) \approx y'(t_i) = f(t_i, y(t_i))$$
**Theorem**

Suppose $f$ is continuous and satisfies a Lipschitz condition with constant $L$ on

$$D = \{(t, y) \mid a \leq t \leq b, -\infty < y < \infty\}$$

and that a constant $M$ exists with

$$|y''(t)| \leq M, \quad \text{for all } t \in [a, b].$$

Let $y(t)$ denote the unique solution to the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

and $w_0, w_1, \ldots, w_n$ as in Euler's method. Then

$$|y(t_i) - w_i| \leq \frac{hM}{2L} \left[ e^{L(t_i-a)} - 1 \right].$$
The difference method

\[ w_0 = \alpha \]
\[ w_{i+1} = w_i + h\phi(t_i, w_i) \]

has local truncation error

\[ \tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\phi(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i), \]

for each \( i = 0, 1, \ldots, N - 1 \).
Consider initial-value problem

\[ y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha. \]

Expand \( y(t) \) in \( n \)th Taylor polynomial about \( t_i \), evaluated at \( t_{i+1} \):

\[
y(t_{i+1}) = y(t_i) + h y'(t_i) + \frac{h^2}{2} y''(t_i) + \cdots
+ \frac{h^n}{n!} y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i)\]

\[
= y(t_i) + h f(t_i, y(t_i)) + \frac{h^2}{2} f'(t_i, y(t_i)) + \cdots
+ \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))
\]

for some \( \xi_i \in (t_i, t_{i+1}) \). Delete remainder term to obtain the Taylor method of order \( n \).
Taylor Method of Order $n$

\[ w_0 = \alpha \]
\[ w_{i+1} = w_i + hT^{(n)}(t_i, w_i), \quad i = 0, 1, \ldots, N - 1 \]

where

\[ T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{(n-1)}}{n!} f^{(n-1)}(t_i, w_i) \]
Theorem

If Taylor’s method of order $n$ is used to approximate the solution to

$$y'(t) = f(t, y(t)), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

with step size $h$ and if $y \in C^{n+1}[a, b]$, then the local truncation error is $O(h^n)$. 
Taylor’s Theorem in Two Variables

Theorem

Suppose $f(t, y)$ and partial derivatives up to order $n + 1$ continuous on $D = \{(t, y) \mid a \leq t \leq b, c \leq y \leq d\}$, let $(t_0, y_0) \in D$. For $(t, y) \in D$, there is $\xi \in [t, t_0]$ and $\mu \in [y, y_0]$ with

$$f(t, y) = P_n(t, y) + R_n(t, y)$$

$$P_n(t, y) = f(t_0, y_0) + \left[ (t - t_0) \frac{\partial f}{\partial t}(t_0, y_0) + (y - y_0) \frac{\partial f}{\partial y}(t_0, y_0) \right]$$

$$+ \left[ \frac{(t - t_0)^2}{2} \frac{\partial^2 f}{\partial t^2}(t_0, y_0) + (t - t_0)(y - y_0) \frac{\partial^2 f}{\partial t \partial y}(t_0, y_0) \right]$$

$$+ \left[ \frac{(y - y_0)^2}{2} \frac{\partial^2 f}{\partial y^2}(t_0, y_0) \right] + \cdots$$

$$+ \left[ \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} (t - t_0)^{n-j} (y - y_0)^j \frac{\partial^n f}{\partial t^{n-j} \partial y^j}(t_0, y_0) \right]$$
Theorem (cont’d)

\[ R_n(t, y) = \frac{1}{(n + 1)!} \sum_{j=0}^{n+1} \binom{n+1}{j} (t - t_0)^{n+1-j} (y - y_0)^j. \]

\[ \cdot \frac{\partial^{n+1} f}{\partial t^{n+1-j} \partial y^j} (\xi, \mu) \]

\( P_n(t, y) \) is the \( n \)th Taylor polynomial in two variables.
Runge-Kutta Methods

- Obtain high-order accuracy of Taylor methods without knowledges of derivatives of \( f \)
- Determine \( a_1, \alpha_1, \beta_1 \) such that

\[
a_1 f(t + \alpha_1, y + \beta_1) \approx f(t, y) + \frac{h}{2} f'(t, y) = T^{(2)}(t, y).
\]

with \( O(h^2) \) error.
- Since

\[
f'(t, y) = \frac{df}{dt}(t, y) = \frac{\partial f}{\partial t}(t, y) + \frac{\partial f}{\partial y}(t, y) \cdot y'(t)
\]

and \( y'(t) = f(t, y) \), we have

\[
T^{(2)}(t, y) = f(t, y) + \frac{h}{2} \frac{\partial f}{\partial t}(t, y) + \frac{h}{2} \frac{\partial f}{\partial y}(t, y) \cdot f(t, y)
\]
Expand \( f(t + \alpha_1, y + \beta_1) \) in 1st degree Taylor polynomial:

\[
a_1 f(t + \alpha_1, y + \beta_1) = a_1 f(t, y) + a_1 \alpha_1 \frac{\partial f}{\partial t}(t, y) + a_1 \beta_1 \frac{\partial f}{\partial y}(t, y) + a_1 \cdot R_1(t + \alpha_1, y + \beta_1)
\]

Matching coefficients gives

\[
a_1 = 1 \quad a_1 \alpha_1 = \frac{h}{2}, \quad a_1 \beta_1 = \frac{h}{2} f(t, y)
\]

with unique solution

\[
a_1 = 1, \quad \alpha_1 = \frac{h}{2}, \quad \beta_1 = \frac{h}{2} f(t, y)
\]
Runge-Kutta Methods

This gives

\[ T^{(2)}(t, y) = f \left( t + \frac{h}{2}, y + \frac{h}{2} f(t, y) \right) - R_1 \left( t + \frac{h}{2}, y + \frac{h}{2} f(t, y) \right) \]

with \( R_1(\cdot, \cdot) = O(h^2) \)

Midpoint Method

\[ w_0 = \alpha, \]
\[ w_{i+1} = w_i + hf \left( t + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i) \right), \quad i = 0, 1, \ldots, N - 1 \]

Local truncation error of order two.
Runge-Kutta Methods

Runge-Kutta Order Four

\[ w_0 = \alpha \]
\[ k_1 = hf(t_i, w_i) \]
\[ k_2 = hf \left( t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1 \right) \]
\[ k_3 = hf \left( t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2 \right) \]
\[ k_4 = hf(t_{i+1}, w_i + k_3) \]
\[ w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \]

Local truncation error \( O(h^4) \)
function \[t, w\] = rk4(f, a, b, alpha, N)
% Solve ODE \(y'(t) = f(t, y(t))\) using Runge–Kutta 4.

\[h = (b-a) / N;\]
\[t = (a:h:b)';\]
\[w = zeros(N+1, length(alpha));\]
\[w(1,:) = alpha(:)';\]
\[for \ i = 1:N\]
\[\quad k1 = h*f(t(i), \ w(i,:));\]
\[\quad k2 = h*f(t(i) + h/2, w(i,:) + k1/2);\]
\[\quad k3 = h*f(t(i) + h/2, w(i,:) + k2/2);\]
\[\quad k4 = h*f(t(i) + h, w(i,:) + k3);\]
\[\quad w(i+1,:) = w(i,:) + (k1 + 2*k2 + 2*k3 + k4)/6;\]
\[end\]
An \textit{m-step multistep method} for solving the initial-value problem

\[ y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha, \]

has a difference equation for approximate \( w_{i+1} \) at \( t_{i+1} \):

\[
\begin{align*}
w_{i+1} &= a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0w_{i+1-m} \\
&\quad + h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots \\
&\quad + b_0 f(t_{i+1-m}, w_{i+1-m})],
\end{align*}
\]

where \( h = (b - a)/N \), and starting values are specified:

\[ w_0 = \alpha, \quad w_1 = \alpha_1, \quad \ldots, \quad w_{m-1} = \alpha_{m-1} \]

\textit{Explicit} method if \( b_m = 0 \), \textit{implicit} method if \( b_m \neq 0 \).
Multistep Methods

Fourth-Order Adams-Bashforth Technique

\[
\begin{align*}
    w_0 &= \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3, \\
    w_{i+1} &= w_i + \frac{h}{24} \left[ 55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) \\
                  &\quad + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3}) \right]
\end{align*}
\]

Fourth-Order Adams-Moulton Technique

\[
\begin{align*}
    w_0 &= \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \\
    w_{i+1} &= w_i + \frac{h}{24} \left[ 9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) \\
                  &\quad - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2}) \right]
\end{align*}
\]
Integrate the initial-value problem

\[ y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha \]

over \([t_i, t_{i+1}]\):

\[ y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t)) \, dt \]

Replace \(f\) by polynomial \(P(t)\) interpolating \((t_0, w_0), \ldots, (t_i, w_i)\), and approximate \(y(t_i) \approx w_i\):

\[ y(t_{i+1}) \approx w_i + \int_{t_i}^{t_{i+1}} P(t) \, dt \]
Adams-Bashforth explicit $m$-step: Newton backward-difference polynomial through

$$(t_i, f(t_i, y(t_i))), \ldots, (t_{i+1-m}, f(t_{i+1-m}, y(t_{i+1-m}))).$$

\[
\int_{t_i}^{t_{i+1}} f(t, y(t)) \, dt \approx \int_{t_i}^{t_{i+1}} \sum_{k=0}^{m-1} (-1)^k \binom{-s}{k} \nabla^k f(t_i, y(t_i)) \, dt \\
= \sum_{k=0}^{m-1} \nabla^k f(t_i, y(t_i)) h (-1)^k \int_0^1 \binom{-s}{k} \, ds
\]

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<th>1</th>
<th>2</th>
<th>3</th>
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<td>$\frac{3}{8}$</td>
<td>$\frac{251}{720}$</td>
<td>$\frac{95}{288}$</td>
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Derivation of Multistep Methods

Three-step Adams-Bashforth:

\[ y(t_{i+1}) \approx y(t_i) + h \left[ f(t_i, y(t_i)) + \frac{1}{2} \nabla f(t_i, y(t_i)) + \frac{5}{12} \nabla^2 f(t_i, y(t_i)) \right] \]

\[ = y(t_i) + \frac{h}{12} \left[ 23 f(t_i, y(t_i)) - 16 f(t_{i-1}, y(t_{i-1})) + 5 f(t_{i-2}, y(t_{i-2})) \right] \]

The method is:

\[ w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \]

\[ w_{i+1} = w_i + \frac{h}{12} \left[ 23 f(t_i, w_i) - 16 f(t_{i-1}, w_{i-1}) + 5 f(t_{i-2}, w_{i-2}) \right] \]
Local Truncation Error

**Definition**

If $y(t)$ solves

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

and

$$w_{i+1} = a_{m-1}w_i + \cdots + a_0 w_{i+1-m}$$
$$+ h[b_m f(t_{i+1}, w_{i+1}) + \cdots + b_0 f(t_{i+1-m}, w_{i+1-m})],$$

the *local truncation error* is

$$\tau_{i+1}(h) = \frac{y(t_{i+1}) - a_{m-1}y(t_i) - \cdots - a_0 y(t_{i+1-m})}{h}$$
$$- [b_m f(t_{i+1}, y(t_{i+1})) + \cdots + b_0 f(t_{i+1-m}, y(t_{i+1-m}))].$$
An \textit{mth-order system} of first-order initial-value problems has the form

\begin{align*}
\frac{du_1}{dt}(t) &= f_1(t, u_1, u_2, \ldots, u_m), \\
\frac{du_2}{dt}(t) &= f_2(t, u_1, u_2, \ldots, u_m), \\
&\quad \vdots \\
\frac{du_m}{dt}(t) &= f_m(t, u_1, u_2, \ldots, u_m),
\end{align*}

for \(a \leq t \leq b\), with the initial conditions

\begin{align*}
u_1(a) &= \alpha_1, u_2(a) = \alpha_2, \ldots, u_m(a) = \alpha_m.
\end{align*}
Definition

The function \( f(t, y_1, \ldots, y_m) \), defined on the set

\[
D = \{(t, u_1, \ldots, u_m) \mid a \leq t \leq b, -\infty < u_i < \infty, i = 1, 2, \ldots, m\}
\]

is said to satisfy a \textit{Lipschitz condition} on \( D \) in the variables \( u_1, u_2, \ldots, u_m \) if a constant \( L > 0 \) exists with

\[
|f(t, u_1, \ldots, u_m) - f(t, z_1, \ldots, z_m)| \leq L \sum_{j=1}^{m} |u_j - z_j|,
\]

for all \((t, u_1, \ldots, u_m)\) and \((t, z_1, \ldots, z_m)\) in \( D \).
Existence and Uniqueness

**Theorem**

Suppose

\[ D = \{(t, u_1, \ldots, u_m) \mid a \leq t \leq b, -\infty < u_i < \infty, i = 1, 2, \ldots, m\} \]

and let \( f_i(t, u_1, \ldots, u_m) \), for each \( i = 1, 2, \ldots, m \), be continuous on \( D \) and satisfy a Lipschitz condition there. The system of first-order differential equations

\[
\frac{du_k}{dt}(t) = f_k(t, u_1, \ldots, u_m), \quad u_k(a) = \alpha_k, \quad k = 1, \ldots, m
\]

has a unique solution \( u_1(t), \ldots, u_m(t) \) for \( a \leq t \leq b \).

**Fourth order Runge-Kutta for systems**

\[
\begin{align*}
    w_0 &= \alpha \\
    k_1 &= hf(t_i, w_i) \\
    k_2 &= hf(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1) \\
    k_3 &= hf(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2) \\
    k_4 &= hf(t_i + 1, w_i + k_3) \\
    w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{align*}
\]

where \( w_i = (w_{i,1}, \ldots, w_{i,m}) \) is the vector of unknowns.
Definition

A one-step difference-equation with local truncation error $\tau_i(h)$ is said to be \textit{consistent} if

$$\lim_{h \to 0} \max_{1 \leq i \leq N} |\tau_i(h)| = 0$$

Definition

A one-step difference equation is said to be \textit{convergent} if

$$\lim_{h \to 0} \max_{1 \leq i \leq N} |w_i - y(t_i)| = 0,$$

where $y_i = y(t_i)$ is the exact solution and $w_i$ the approximation.
**Theorem**

Suppose the initial-value problem $y' = f(t, y), \ a \leq t \leq b$, $y(a) = \alpha$ is approximated by a one-step difference method in the form $w_0 = \alpha, w_{i+1} = w_i + h\phi(t_i, w_i, h)$. Suppose also that $h_0 > 0$ exists and $\phi(t, w, h)$ is continuous with a Lipschitz condition in $w$ with constant $L$ on $D$, then

$$D = \{(t, w, h) \mid a \leq t \leq b, -\infty < w < \infty, 0 \leq h \leq h_0\}.$$

1. **The method is stable;**

2. **The method is convergent if and only if it is consistent:**

$$\phi(t, y, 0) = f(t, y)$$

3. If $\tau$ exists s.t. $|\tau_i(h)| \leq \tau(h)$ when $0 \leq h \leq h_0$, then

$$|y(t_i) - w_i| \leq \frac{\tau(h)}{L}e^{L(t_i-a)}.$$
Let $\lambda_1, \ldots, \lambda_m$ denote the roots of the characteristic equation

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - \cdots - a_1\lambda - a_0 = 0$$

associated with the multistep method

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0w_{i+1-m} + hF(t_i, h, w_{i+1}, w_i, \ldots, w_{i+1-m}).$$

If $|\lambda_i| \leq 1$ and all roots with absolute value 1 are simple, the method is said to satisfy the root condition.
Stability

Definition

1. Methods that satisfy the root condition and have $\lambda = 1$ as the only root of magnitude one are called strongly stable.
2. Methods that satisfy the root condition and have more than one distinct root with magnitude one are called weakly stable.
3. Methods that do not satisfy the root condition are unstable.

Theorem

A multistep method

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0w_{i+1-m} + hF(t_i, h, w_{i+1}, w_i, \ldots, w_{i+1-m})$$

is stable if and only if it satisfies the root condition. If it is also consistent, then it is stable if and only if it is convergent.
A stiff differential equation is numerically unstable unless the step size is extremely small.

Large derivatives give error terms that are dominating the solution.

Example: The initial-value problem

\[ y' = -30y, \quad 0 \leq t \leq 1.5, \quad y(0) = \frac{1}{3} \]

has exact solution \( y = \frac{1}{3}e^{-30t} \). But RK4 is unstable with step size \( h = 0.1 \).
Consider the simple test equation

\[ y' = \lambda y, \quad y(0) = \alpha, \quad \text{where } \lambda < 0 \]

with solution \( y(t) = \alpha e^{\lambda t} \).

Euler’s method gives \( w_0 = \alpha \) and

\[ w_{j+1} = w_j + h(\lambda w_j) = (1 + h\lambda)w_j = (1 + h\lambda)^{j+1}\alpha. \]

The absolute error is

\[ |y(t_j) - w_j| = \left| e^{jh\lambda} - (1 + h\lambda)^j \right| |\alpha| \]

\[ = \left| (e^{h\lambda})^j - (1 + h\lambda)^j \right| |\alpha| \]

Stability requires \(|1 + h\lambda| < 1\), or \( h < 2/|\lambda| \).
Apply a multistep method to the test equation:

\[ w_{j+1} = a_{m-1}w_j + \cdots + a_0w_{j+1-m} + h\lambda (b_mw_{j+1} + b_{m-1}w_j + \cdots + b_0w_{j+1-m}) \]

or

\[ (1 - h\lambda b_m)w_{j+1} - (a_{m-1} + h\lambda b_{m-1})w_j - \cdots - (a_0 + h\lambda b_0)w_{j+1-m} = 0 \]

Let \( \beta_1, \ldots, \beta_m \) be the zeros of the characteristic polynomial

\[ Q(z, h\lambda) = (1 - h\lambda b_m)z^m - (a_{m-1} + h\lambda b_{m-1})z^{m-1} - \cdots - (a_0 + h\lambda b_0) \]

Then \( c_1, \ldots, c_m \) exist with

\[ w_j = \sum_{k=1}^{m} c_k (\beta_k)^j \]

and \( |\beta_k| < 1 \) is required for stability.
The region $R$ of absolute stability for a one-step method is $R = \{ h\lambda \in \mathbb{C} \mid |Q(h\lambda)| < 1 \}$, and for a multistep method, it is $R = \{ h\lambda \in \mathbb{C} \mid |\beta_k| < 1, \text{ for all zeros } \beta_k \text{ of } Q(z, h\lambda) \}$.

A numerical method is said to be $A$-stable if its region $R$ of absolute stability contains the entire left half-plane.