Chapter 6
Direct Methods for Solving Linear Systems

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Direct Methods for Linear Systems

Consider solving a linear system of the form:

$$
\begin{aligned}
E_1 &: a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\
E_2 &: a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\
&\qquad\qquad\qquad\qquad\vdots \\
E_n &: a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n,
\end{aligned}
$$

for $x_1, \dots, x_n$. *Direct methods* give an answer in a fixed number of steps, subject only to round-off errors.

We use three row operations to simplify the linear system:

1. Multiply Eq. E_i by $\lambda \neq 0$: $(\lambda E_i) \to (E_i)$
2. Multiply Eq. E_j by λ and add to Eq. E_i: $(E_i + \lambda E_j) \to (E_i)$
3. Exchange Eq. E_i and Eq. E_j: $(E_i) \leftrightarrow (E_j)$

Gaussian Elimination

Gaussian Elimination with Backward Substitution

- Reduce a linear system to *triangular form* by introducing zeros using the row operations $(E_i + \lambda E_j) \rightarrow (E_i)$
- Solve the triangular form using *backward-substitution*

Row Exchanges

- If a *pivot element* on the diagonal is zero, the reduction to triangular form fails
- Find a nonzero element below the diagonal and exchange the two rows

Definition

An $n \times m$ *matrix* is a rectangular array of elements with n rows and m columns in which both value and position of an element is important

Operation Counts

- Count the number of arithmetic operations performed
- Use the formulas

$$\sum_{j=1}^{m} j = \frac{m(m+1)}{2}, \quad \sum_{j=1}^{m} j^2 = \frac{m(m+1)(2m+1)}{6}$$

Reduction to Triangular Form

Multiplications/divisions:

$$\sum_{i=1}^{n-1} (n-i)(n-i+2) = \cdots = \frac{2n^3+3n^2-5n}{6}$$

Additions/subtractions:

$$\sum_{i=1}^{n-1} (n-i)(n-i+1) = \cdots = \frac{n^3-n}{3}$$

Operation Counts

Backward Substitution

Multiplications/divisions:

$$1 + \sum_{i=1}^{n-1} ((n-i)+1) = \frac{n^2+n}{2}$$

Additions/subtractions:

$$\sum_{i=1}^{n-1} ((n-i-1)+1) = \frac{n^2-n}{2}$$

Operation Counts

Gaussian Elimination Total Operation Count

Multiplications/divisions:

$$\frac{n^3}{3} + n^2 - \frac{n}{3}$$

Additions/subtractions:

$$\frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6}$$

Partial Pivoting

- In Gaussian elimination, if a pivot element $a_{kk}^{(k)}$ is small compared to an element $a_{jk}^{(k)}$ below, the multiplier

$$m_{jk} = \frac{a_{jk}^{(k)}}{a_{kk}^{(k)}}$$

will be large, resulting in round-off errors
- *Partial pivoting* finds the smallest $p \geq k$ such that

$$|a_{pk}^{(k)}| = \max_{k \leq i \leq n} |a_{ik}^{(k)}|$$

and interchanges the rows $(E_k) \leftrightarrow (E_p)$

Scaled Partial Pivoting

- If there are large variations in magnitude of the elements within a row, *scaled partial pivoting* can be used
- Define a scale factor s_i for each row

$$s_i = \max_{1 \le j \le n} |a_{ij}|$$

- At step i, find p such that

$$\frac{|a_{pi}|}{s_p} = \max_{i \le k \le n} \frac{|a_{ki}|}{s_k}$$

and interchange the rows $(E_i) \leftrightarrow (E_p)$

Definition

Two matrices A and B are *equal* if they have the same number of rows and columns $n \times m$ and if $a_{ij} = b_{ij}$.

Definition

If A and B are $n \times m$ matrices, the *sum* $A + B$ is the $n \times m$ matrix with entries $a_{ij} + b_{ij}$.

Definition

If A is $n \times m$ and λ a real number, the *scalar multiplication* λA is the $n \times m$ matrix with entries λa_{ij}.

Properties

Theorem

Let A, B, C be $n \times m$ matrices, λ, μ real numbers.

(a) $A + B = B + A$
(b) $(A + B) + C = A + (B + C)$
(c) $A + 0 = 0 + A = A$
(d) $A + (-A) = -A + A = 0$
(e) $\lambda(A + B) = \lambda A + \lambda B$
(f) $(\lambda + \mu)A = \lambda A + \mu A$
(g) $\lambda(\mu A) = (\lambda\mu)A$
(h) $1A = A$

Matrix Multiplication

Definition

Let A be $n \times m$ and B be $m \times p$. The *matrix product* $C = AB$ is the $n \times p$ matrix with entries

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{im} b_{mj}$$

Special Matrices

Definition

- A *square* matrix has $m = n$
- A *diagonal* matrix $D = [d_{ij}]$ is square with $d_{ij} = 0$ when $i \neq j$
- The *identity matrix of order* n, $I_n = [\delta_{ij}]$, is diagonal with

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Definition

- An *upper-triangular* $n \times n$ matrix $U = [u_{ij}]$ has

$$u_{ij} = 0, \qquad \text{if } i = j+1, \ldots, n.$$

- A *lower-triangular* $n \times n$ matrix $L = [l_{ij}]$ has

$$l_{ij} = 0, \qquad \text{if } i = 1, \ldots, j-1.$$

Properties

Theorem

Let A be $n \times m$, B be $m \times k$, C be $k \times p$, D be $m \times k$, and λ a real number.

(a) $A(BC) = (AB)C$
(b) $A(B + D) = AB + AD$
(c) $I_m B = B$ and $BI_k = B$
(d) $\lambda(AB) = (\lambda A)B = A(\lambda B)$

Matrix Inversion

Definition

- An $n \times n$ matrix A is *nonsingular* or *invertible* if $n \times n$ A^{-1} exists with $AA^{-1} = A^{-1}A = I$
- The matrix A^{-1} is called the *inverse* of A
- A matrix without an inverse is called *singular* or *noninvertible*

Theorem

For any nonsingular $n \times n$ matrix A,

(a) A^{-1} is unique
(b) A^{-1} is nonsingular and $(A^{-1})^{-1} = A$
(c) If B is nonsingular $n \times n$, then $(AB)^{-1} = B^{-1}A^{-1}$

Matrix Transpose

Definition

- The *transpose* of $n \times m$ $A = [a_{ij}]$ is $m \times n$ $A^t = [a_{ji}]$
- A square matrix A is called *symmetric* if $A = A^t$

Theorem

(a) $(A^t)^t = A$
(b) $(A + B)^t = A^t + B^t$
(c) $(AB)^t = B^t A^t$
(d) if A^{-1} exists, then $(A^{-1})^t = (A^t)^{-1}$

Determinants

Definition

(a) If $A = [a]$ is a 1×1 matrix, then $\det A = a$

(b) If A is $n \times n$, the *minor* M_{ij} is the determinant of the $(n-1) \times (n-1)$ submatrix deleting row i and column j of A

(c) The *cofactor* A_{ij} associated with M_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$

(d) The *determinant* of $n \times n$ matrix A for $n > 1$ is

$$\det A = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

or

$$\det A = \sum_{i=1}^{n} a_{ij} A_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

Properties

Theorem

(a) If any row or column of A has all zeros, then $\det A = 0$

(b) If A has two rows or two columns equal, then $\det A = 0$

(c) If $\tilde{A}$ comes from $(E_i) \leftrightarrow (E_j)$ on A, then $\det \tilde{A} = -\det A$

(d) If $\tilde{A}$ comes from $(\lambda E_i) \leftrightarrow (E_i)$ on A, then $\det \tilde{A} = \lambda \det A$

(e) If $\tilde{A}$ comes from $(E_i + \lambda E_j) \leftrightarrow (E_i)$ on A, with $i \neq j$, then $\det \tilde{A} = \det A$

(f) If B is also $n \times n$, then $\det AB = \det A \det B$

(g) $\det A^t = \det A$

(h) When A^{-1} exists, $\det A^{-1} = (\det A)^{-1}$

(i) If A is upper/lower triangular or diagonal, then $\det A = \prod_{i=1}^{n} a_{ii}$

Theorem

The following statements are equivalent for any $n \times n$ matrix A:

(a) The equation $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$

(b) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$

(c) The matrix A is nonsingular; that is, A^{-1} exists

(d) $\det A \neq 0$

(e) Gaussian elimination with row interchanges can be performed on the system $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b}$

LU Factorization

The k*th Gaussian transformation matrix* is defined by

$$M^{(k)} = \begin{bmatrix} 1 & 0 & & \cdots & & \cdots & 0 \\ 0 & \ddots & \ddots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & 0 & \ddots & \ddots & & \vdots \\ \vdots & & \vdots & -m_{k+1,k} & \ddots & \ddots & \vdots \\ \vdots & & \vdots & \vdots & 0 & \ddots & \vdots \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -m_{n,k} & 0 & \cdots & 0 & 1 \end{bmatrix}$$

LU Factorization

Gaussian elimination can be written as

$$A^{(n)} = M^{(n-1)} \cdots M^{(1)} A = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1,n}^{(n-1)} \\ 0 & \cdots & 0 & a_{nn}^{(n)} \end{bmatrix}$$

Reversing the elimination steps gives the inverses:

$$L^{(k)} = [M^{(k)}]^{-1} = \begin{bmatrix} 1 & 0 & & \cdots & & \cdots & & 0 \\ 0 & \ddots & \ddots & & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ \vdots & & 0 & \ddots & \ddots & & & \vdots \\ \vdots & & \vdots & m_{k+1,k} & \ddots & \ddots & & \vdots \\ \vdots & & \vdots & \vdots & 0 & \ddots & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & m_{n,k} & 0 & \cdots & 0 & 1 \end{bmatrix}$$

and we have

$$\begin{aligned} LU &= L^{(1)} \cdots L^{(n-1)} \cdots M^{(n-1)} \cdots M^{(1)} A \\ &= [M^{(1)}]^{-1} \cdots [M^{(n-1)}]^{-1} \cdots M^{(n-1)} \cdots M^{(1)} A = A \end{aligned}$$

LU Factorization

Theorem

If Gaussian elimination can be performed on the linear system $A\mathbf{x} = \mathbf{b}$ without row interchanges, A can be factored into the product of lower-triangular L and upper-triangular U as $A = LU$, where $m_{ji} = a_{ji}^{(i)}/a_{ii}^{(i)}$:

$$U = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1,n}^{(n-1)} \\ 0 & \cdots & 0 & a_{nn}^{(n)} \end{bmatrix}, \; L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_{21} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ m_{n1} & \cdots & m_{n,n-1} & 1 \end{bmatrix}$$

Permutation Matrices

Suppose $k_1, \ldots, k_n$ is a permutation of $1, \ldots, n$. The permutation matrix $P = (p_{ij})$ is defined by

$$p_{ij} = \begin{cases} 1, & \text{if } j = k_i, \\ 0, & \text{otherwise.} \end{cases}$$

(i) PA permutes the rows of A:

$$PA = \begin{bmatrix} a_{k_1 1} & \cdots & a_{k_1 n} \\ \vdots & \ddots & \vdots \\ a_{k_n 1} & \cdots & a_{k_n n} \end{bmatrix}$$

(ii) P^{-1} exists and $P^{-1} = P^t$

Gaussian elimination with row interchanges then becomes:

$$A = P^{-1}LU = (P^t L)U$$

Diagonally Dominant Matrices

Definition

The $n \times n$ matrix A is said to be *strictly diagonally dominant* when

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

Theorem

A strictly diagonally dominant matrix A is nonsingular, Gaussian elimination can be performed on $A\mathbf{x} = \mathbf{b}$ without row interchanges, and the computations will be stable.

Positive Definite Matrices

Definition

A matrix A is *positive definite* if it is symmetric and if $\mathbf{x}^t A\mathbf{x} > 0$ for every $\mathbf{x} \neq \mathbf{0}$.

Theorem

If A is an $n \times n$ positive definite matrix, then

(a) A has an inverse
(b) $a_{ii} > 0$
(c) $\max_{1\leq k,j\leq n} |a_{kj}| \leq \max_{1\leq i\leq n} |a_{ii}|$
(d) $(a_{ij})^2 < a_{ii}a_{jj}$ for $i \neq j$

Principal Submatrices

Definition

A *leading principal submatrix* of a matrix A is a matrix of the form

$$A_k = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix}$$

for some $1 \leq k \leq n$.

Theorem

A symmetric matrix A is positive definite if and only if each of its leading principal submatrices has a positive determinant.

SPD and Gaussian Elimination

Theorem

The symmetric matrix A is positive definite if and only if Gaussian elimination without row interchanges can be done on $A\mathbf{x} = \mathbf{b}$ with all pivot elements positive, and the computations are then stable.

Corollary

The matrix A is positive definite if and only if it can be factored $A = LDL^t$ where L is lower triangular with 1's on its diagonal and D is diagonal with positive diagonal entries.

Corollary

The matrix A is positive definite if and only if it can be factored $A = LL^t$, where L is lower triangular with nonzero diagonal entries.

Band Matrices

Definition

An $n \times n$ matrix is called a *band matrix* if p, q exist with $1 < p, q < n$ and $a_{ij} = 0$ when $p \leq j - i$ or $q \leq i - j$. The *bandwidth* is $w = p + q - 1$.

A *tridiagonal* matrix has $p = q = 2$ and bandwidth 3.

Theorem

Suppose $A = [a_{ij}]$ is tridiagonal with $a_{i,i-1}a_{i,i+1} \neq 0$. If $|a_{11}| > |a_{12}|$, $|a_{ii}| \geq |a_{i,i-1}| + |a_{i,i+1}|$, and $|a_{nn}| > |a_{n,n-1}|$, then A is nonsingular.