Chapter 6
Direct Methods for Solving Linear Systems

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Consider solving a linear system of the form:

\[ E_1 : a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \]
\[ E_2 : a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \]
\[ \vdots \]
\[ E_n : a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n, \]

for \( x_1, \ldots, x_n \). Direct methods give an answer in a fixed number of steps, subject only to round-off errors.

We use three row operations to simplify the linear system:

1. Multiply Eq. \( E_i \) by \( \lambda \neq 0 \): \((\lambda E_i) \rightarrow (E_i)\)
2. Multiply Eq. \( E_j \) by \( \lambda \) and add to Eq. \( E_i \): \((E_i + \lambda E_j) \rightarrow (E_i)\)
3. Exchange Eq. \( E_i \) and Eq. \( E_j \): \((E_i) \leftrightarrow (E_j)\)
Gaussian Elimination

Gaussian Elimination with Backward Substitution

- Reduce a linear system to *triangular form* by introducing zeros using the row operations \((E_i + \lambda E_j) \rightarrow (E_i)\)
- Solve the triangular form using *backward-substitution*

Row Exchanges

- If a *pivot element* on the diagonal is zero, the reduction to triangular form fails
- Find a nonzero element below the diagonal and exchange the two rows

Definition

An \(n \times m \) *matrix* is a rectangular array of elements with \(n\) rows and \(m\) columns in which both value and position of an element is important
Operation Counts

- Count the number of arithmetic operations performed
- Use the formulas

\[ \sum_{j=1}^{m} j = \frac{m(m + 1)}{2}, \quad \sum_{j=1}^{m} j^2 = \frac{m(m + 1)(2m + 1)}{6} \]

Reduction to Triangular Form

Multiplications/divisions:

\[ \sum_{i=1}^{n-1} (n - i)(n - i + 2) = \cdots = \frac{2n^3 + 3n^2 - 5n}{6} \]

Additions/subtractions:

\[ \sum_{i=1}^{n-1} (n - i)(n - i + 1) = \cdots = \frac{n^3 - n}{3} \]
Operation Counts

Backward Substitution

Multiplications/divisions:

\[ 1 + \sum_{i=1}^{n-1} ((n - i) + 1) = \frac{n^2 + n}{2} \]

Additions/subtractions:

\[ \sum_{i=1}^{n-1} ((n - i - 1) + 1) = \frac{n^2 - n}{2} \]
Gaussian Elimination Total Operation Count

Multiplications/divisions:

\[ \frac{n^3}{3} + n^2 - \frac{n}{3} \]

Additions/subtractions:

\[ \frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6} \]
In Gaussian elimination, if a pivot element $a_{kk}^{(k)}$ is small compared to an element $a_{jk}^{(k)}$ below, the multiplier

$$m_{jk} = \frac{a_{jk}^{(k)}}{a_{kk}^{(k)}}$$

will be large, resulting in round-off errors. **Partial pivoting** finds the smallest $p \geq k$ such that

$$|a_{pk}^{(k)}| = \max_{k \leq i \leq n} |a_{ik}^{(k)}|$$

and interchanges the rows $(E_k) \leftrightarrow (E_p)$
If there are large variations in magnitude of the elements within a row, *scaled partial pivoting* can be used.

Define a scale factor $s_i$ for each row

$$s_i = \max_{1 \leq j \leq n} |a_{ij}|$$

At step $i$, find $p$ such that

$$\frac{|a_{pi}|}{s_p} = \max_{i \leq k \leq n} \frac{|a_{ki}|}{s_k}$$

and interchange the rows $(E_i) \leftrightarrow (E_p)$
**Definition**

Two matrices $A$ and $B$ are *equal* if they have the same number of rows and columns $n \times m$ and if $a_{ij} = b_{ij}$.

**Definition**

If $A$ and $B$ are $n \times m$ matrices, the *sum* $A + B$ is the $n \times m$ matrix with entries $a_{ij} + b_{ij}$.

**Definition**

If $A$ is $n \times m$ and $\lambda$ a real number, the *scalar multiplication* $\lambda A$ is the $n \times m$ matrix with entries $\lambda a_{ij}$. 

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Linear Algebra
Let $A, B, C$ be $n \times m$ matrices, $\lambda, \mu$ real numbers.

(a) $A + B = B + A$
(b) $(A + B) + C = A + (B + C)$
(c) $A + 0 = 0 + A = A$
(d) $A + (-A) = -A + A = 0$
(e) $\lambda(A + B) = \lambda A + \lambda B$
(f) $(\lambda + \mu)A = \lambda A + \mu A$
(g) $\lambda(\mu A) = (\lambda\mu)A$
(h) $1A = A$
Matrix Multiplication

Definition

Let $A$ be $n \times m$ and $B$ be $m \times p$. The matrix product $C = AB$ is the $n \times p$ matrix with entries

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{im} b_{mj}$$
**Definition**

- A *square* matrix has $m = n$.
- A *diagonal* matrix $D = [d_{ij}]$ is square with $d_{ij} = 0$ when $i \neq j$.
- The *identity matrix of order* $n$, $I_n = [\delta_{ij}]$, is diagonal with

$$
\delta_{ij} = \begin{cases} 
1, & \text{if } i = j, \\
0, & \text{if } i \neq j.
\end{cases}
$$

**Definition**

- An *upper-triangular* $n \times n$ matrix $U = [u_{ij}]$ has

$$
u_{ij} = 0, \quad \text{if } i = j + 1, \ldots, n.
$$

- A *lower-triangular* $n \times n$ matrix $L = [l_{ij}]$ has

$$
l_{ij} = 0, \quad \text{if } i = 1, \ldots, j - 1.
$$
Let $A$ be $n \times m$, $B$ be $m \times k$, $C$ be $k \times p$, $D$ be $m \times k$, and $\lambda$ a real number.

(a) $A(BC) = (AB)C$
(b) $A(B + D) = AB + AD$
(c) $I_mB = B$ and $BI_k = B$
(d) $\lambda(AB) = (\lambda A)B = A(\lambda B)$
Matrix Inversion

**Definition**

- An \( n \times n \) matrix \( A \) is **nonsingular** or **invertible** if \( n \times n \) \( A^{-1} \) exists with \( AA^{-1} = A^{-1}A = I \)
- The matrix \( A^{-1} \) is called the **inverse** of \( A \)
- A matrix without an inverse is called **singular** or **noninvertible**

**Theorem**

For any nonsingular \( n \times n \) matrix \( A \),

(a) \( A^{-1} \) is unique

(b) \( A^{-1} \) is nonsingular and \( (A^{-1})^{-1} = A \)

(c) If \( B \) is nonsingular \( n \times n \), then \( (AB)^{-1} = B^{-1}A^{-1} \)
Matrix Transpose

**Definition**
- The *transpose* of $n \times m$ \( A = [a_{ij}] \) is $m \times n$ \( A^t = [a_{ji}] \)
- A square matrix $A$ is called *symmetric* if $A = A^t$

**Theorem**

(a) \((A^t)^t = A\)
(b) \((A + B)^t = A^t + B^t\)
(c) \((AB)^t = B^t A^t\)
(d) if $A^{-1}$ exists, then \((A^{-1})^t = (A^t)^{-1}\)
Determinants

**Definition**

(a) If $A = [a]$ is a $1 \times 1$ matrix, then $\det A = a$

(b) If $A$ is $n \times n$, the *minor* $M_{ij}$ is the determinant of the $(n-1) \times (n-1)$ submatrix deleting row $i$ and column $j$ of $A$

(c) The *cofactor* $A_{ij}$ associated with $M_{ij}$ is $A_{ij} = (-1)^{i+j}M_{ij}$

(d) The *determinant* of $n \times n$ matrix $A$ for $n > 1$ is

$$
\det A = \sum_{j=1}^{n} a_{ij}A_{ij} = \sum_{j=1}^{n} (-1)^{i+j}a_{ij}M_{ij}
$$

or

$$
\det A = \sum_{i=1}^{n} a_{ij}A_{ij} = \sum_{i=1}^{n} (-1)^{i+j}a_{ij}M_{ij}
$$
Properties

Theorem

(a) If any row or column of \( A \) has all zeros, then \( \det A = 0 \)
(b) If \( A \) has two rows or two columns equal, then \( \det A = 0 \)
(c) If \( \tilde{A} \) comes from \( (E_i) \leftrightarrow (E_j) \) on \( A \), then \( \det \tilde{A} = -\det A \)
(d) If \( \tilde{A} \) comes from \( (\lambda E_i) \leftrightarrow (E_i) \) on \( A \), then \( \det \tilde{A} = \lambda \det A \)
(e) If \( \tilde{A} \) comes from \( (E_i + \lambda E_j) \leftrightarrow (E_i) \) on \( A \), with \( i \neq j \), then \( \det \tilde{A} = \det A \)
(f) If \( B \) is also \( n \times n \), then \( \det AB = \det A \det B \)
(g) \( \det A^t = \det A \)
(h) When \( A^{-1} \) exists, \( \det A^{-1} = (\det A)^{-1} \)
(i) If \( A \) is upper/lower triangular or diagonal, then \( \det A = \prod_{i=1}^{n} a_{ii} \)
Theorem

The following statements are equivalent for any $n \times n$ matrix $A$:

(a) The equation $Ax = 0$ has the unique solution $x = 0$
(b) The system $Ax = b$ has a unique solution for any $b$
(c) The matrix $A$ is nonsingular; that is, $A^{-1}$ exists
(d) $\det A \neq 0$
(e) Gaussian elimination with row interchanges can be performed on the system $Ax = b$ for any $b$
The $k$th Gaussian transformation matrix is defined by

$$M^{(k)} = \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & & & & \\
\vdots & & \ddots & & & \\
\vdots & & & 0 & \ddots & \\
\vdots & & & & \ddots & \ddots \\
\vdots & \vdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & -m_{k+1,k} & \cdots & \cdots \\
\vdots & \vdots & \cdots & \cdots & 0 & \ddots \\
\vdots & \vdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & -m_{n,k} & 0 & \cdots & 0 & 1
\end{bmatrix}$$
Gaussian elimination can be written as

\[ A^{(n)} = M^{(n-1)} \cdots M^{(1)} A = \]

\[
\begin{bmatrix}
  a^{(1)}_{11} & a^{(1)}_{12} & \cdots & a^{(1)}_{1n} \\
  0 & a^{(2)}_{22} & \ddots & \vdots \\
  \vdots & \ddots & \ddots & a^{(n-1)}_{n-1,n} \\
  0 & \cdots & 0 & a^{(n)}_{nn}
\end{bmatrix}
\]
Reversing the elimination steps gives the inverses:

\[
L^{(k)} = [M^{(k)}]^{-1} = \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & & & & \\
\vdots & \ddots & \ddots & & & \\
\vdots & & \ddots & \ddots & & \\
\vdots & & & \ddots & \ddots & \\
0 & \cdots & \cdots & \cdots & \ddots & m_{k+1,k} \\
\vdots & \cdots & \cdots & \cdots & \ddots & \ddots \\
\vdots & \cdots & \cdots & \cdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & m_{n,k} & 0 & \cdots & 0 & 1
\end{bmatrix}
\]

and we have

\[
LU = L^{(1)} \cdots L^{(n-1)} \cdots M^{(n-1)} \cdots M^{(1)} A \\
= [M^{(1)}]^{-1} \cdots [M^{(n-1)}]^{-1} \cdots M^{(n-1)} \cdots M^{(1)} A = A
\]
Theorem

If Gaussian elimination can be performed on the linear system $A\mathbf{x} = \mathbf{b}$ without row interchanges, $A$ can be factored into the product of lower-triangular $L$ and upper-triangular $U$ as $A = LU$, where $m_{ji} = a_{ji}^{(i)}/a_{ii}^{(i)}$:

$$U = \begin{bmatrix}
a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\
0 & a_{22}^{(2)} & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & a_{n-1,n}^{(n-1)} & a_{nn}^{(n)}
\end{bmatrix}, \quad L = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
m_{21} & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
m_{n1} & \cdots & m_{n,n-1} & 1
\end{bmatrix}$$
Suppose \( k_1, \ldots, k_n \) is a permutation of 1, \( \ldots, n \). The permutation matrix \( P = (p_{ij}) \) is defined by

\[
p_{ij} = \begin{cases} 
1, & \text{if } j = k_i, \\
0, & \text{otherwise}.
\end{cases}
\]

(i) \( PA \) permutes the rows of \( A \):

\[
PA = \begin{bmatrix}
  a_{k_11} & \cdots & a_{k_1n} \\
  \vdots & \ddots & \vdots \\
  a_{k_n1} & \cdots & a_{k nn}
\end{bmatrix}
\]

(ii) \( P^{-1} \) exists and \( P^{-1} = P^t \)

Gaussian elimination with row interchanges then becomes:

\[
A = P^{-1}LU = (P^tL)U
\]
Diagonally Dominant Matrices

**Definition**

The $n \times n$ matrix $A$ is said to be *strictly diagonally dominant* when

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

**Theorem**

A strictly diagonally dominant matrix $A$ is nonsingular, Gaussian elimination can be performed on $Ax = b$ without row interchanges, and the computations will be stable.
Positive Definite Matrices

Definition

A matrix $A$ is *positive definite* if it is symmetric and if $x^t A x > 0$ for every $x \neq 0$.

Theorem

If $A$ is an $n \times n$ positive definite matrix, then

(a) $A$ has an inverse
(b) $a_{ii} > 0$
(c) $\max_{1 \leq k, j \leq n} |a_{kj}| \leq \max_{1 \leq i \leq n} |a_{ii}|$
(d) $(a_{ij})^2 < a_{ii} a_{jj}$ for $i \neq j$
Definition

A leading principal submatrix of a matrix $A$ is a matrix of the form

$$A_k = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1k} \\
    a_{21} & a_{22} & \cdots & a_{2k} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{k1} & a_{k2} & \cdots & a_{kk}
\end{bmatrix}$$

for some $1 \leq k \leq n$.

Theorem

A symmetric matrix $A$ is positive definite if and only if each of its leading principal submatrices has a positive determinant.
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<thead>
<tr>
<th>Theorem</th>
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<tbody>
<tr>
<td>The symmetric matrix $A$ is positive definite if and only if Gaussian elimination without row interchanges can be done on $Ax = b$ with all pivot elements positive, and the computations are then stable.</td>
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<td>The matrix $A$ is positive definite if and only if it can be factored $A = LDL^t$ where $L$ is lower triangular with 1’s on its diagonal and $D$ is diagonal with positive diagonal entries.</td>
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<td>The matrix $A$ is positive definite if and only if it can be factored $A = LL^t$, where $L$ is lower triangular with nonzero diagonal entries.</td>
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Definition

An $n \times n$ matrix is called a band matrix if $p, q$ exist with $1 < p, q < n$ and $a_{ij} = 0$ when $p \leq j - i$ or $q \leq i - j$. The bandwidth is $w = p + q - 1$.

A tridiagonal matrix has $p = q = 2$ and bandwidth 3.

Theorem

Suppose $A = [a_{ij}]$ is tridiagonal with $a_{i,i-1}a_{i,i+1} \neq 0$. If $|a_{11}| > |a_{12}|$, $|a_{ii}| \geq |a_{i,i-1}| + |a_{i,i+1}|$, and $|a_{nn}| > |a_{n,n-1}|$, then $A$ is nonsingular.