Direct Methods for Linear Systems

Consider solving a linear system of the form:

\[ E_1 : a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \]
\[ E_2 : a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \]
\[ \vdots \]
\[ E_n : a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n, \]

for \( x_1, \ldots, x_n \). Direct methods give an answer in a fixed number of steps, subject only to round-off errors.

We use three row operations to simplify the linear system:

1. Multiply Eq. \( E_i \) by \( \lambda \neq 0 \): \( (\lambda E_i) \rightarrow (E_i) \)
2. Multiply Eq. \( E_j \) by \( \lambda \) and add to Eq. \( E_i \): \( (E_i + \lambda E_j) \rightarrow (E_i) \)
3. Exchange Eq. \( E_i \) and Eq. \( E_j \): \( (E_i) \leftrightarrow (E_j) \)
Gaussian Elimination

Gaussian Elimination with Backward Substitution
- Reduce a linear system to *triangular form* by introducing zeros using the row operations $(E_i + \lambda E_j) \rightarrow (E_i)$
- Solve the triangular form using *backward-substitution*

Row Exchanges
- If a *pivot element* on the diagonal is zero, the reduction to triangular form fails
- Find a nonzero element below the diagonal and exchange the two rows

Definition
- An $n \times m$ *matrix* is a rectangular array of elements with $n$ rows and $m$ columns in which both value and position of an element is important
Operation Counts

- Count the number of arithmetic operations performed
- Use the formulas

\[
\sum_{j=1}^{m} j = \frac{m(m + 1)}{2}, \quad \sum_{j=1}^{m} j^2 = \frac{m(m + 1)(2m + 1)}{6}
\]

Reduction to Triangular Form

Multiplications/divisions:

\[
\sum_{i=1}^{n-1} (n - i)(n - i + 2) = \cdots = \frac{2n^3 + 3n^2 - 5n}{6}
\]

Additions/subtractions:

\[
\sum_{i=1}^{n-1} (n - i)(n - i + 1) = \cdots = \frac{n^3 + 3n^2 - 5n}{6}
\]
Backward Substitution

Multiplications/divisions:

\[ 1 + \sum_{i=1}^{n-1} ((n - i) + 1) = \frac{n^2 + n}{2} \]

Additions/subtractions:

\[ \sum_{i=1}^{n-1} ((n - i - 1) + 1) = \frac{n^2 - n}{2} \]
Operation Counts

Gaussian Elimination Total Operation Count

Multiplications/divisions:

\[
\frac{n^3}{3} + n^2 - \frac{n}{3}
\]

Additions/subtractions:

\[
\frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6}
\]
In Gaussian elimination, if a pivot element $a_{kk}^{(k)}$ is small compared to an element $a_{jk}^{(k)}$ below, the multiplier

$$m_{jk} = \frac{a_{jk}^{(k)}}{a_{kk}^{(k)}}$$

will be large, resulting in round-off errors.

**Partial pivoting** finds the smallest $p \geq k$ such that

$$|a_{pk}^{(k)}| = \max_{k \leq i \leq n} |a_{ik}^{(k)}|$$

and interchanges the rows $(E_k) \leftrightarrow (E_p)$.
If there are large variations in magnitude of the elements within a row, *scaled partial pivoting* can be used.

Define a scale factor $s_i$ for each row

$$s_i = \max_{1 \leq j \leq n} |a_{ij}|$$

At step $i$, find $p$ such that

$$\frac{|a_{pi}|}{s_p} = \max_{i \leq k \leq n} \frac{|a_{ki}|}{s_k}$$

and interchange the rows $(E_i) \leftrightarrow (E_p)$.
Definition

Two matrices $A$ and $B$ are equal if they have the same number of rows and columns $n \times m$ and if $a_{ij} = b_{ij}$.

Definition

If $A$ and $B$ are $n \times m$ matrices, the sum $A + B$ is the $n \times m$ matrix with entries $a_{ij} + b_{ij}$.

Definition

If $A$ is $n \times m$ and $\lambda$ a real number, the scalar multiplication $\lambda A$ is the $n \times m$ matrix with entries $\lambda a_{ij}$. 
Properties

**Theorem**

Let $A, B, C$ be $n \times m$ matrices, $\lambda, \mu$ real numbers.

(a) $A + B = B + A$

(b) $(A + B) + C = A + (B + C)$

(c) $A + 0 = 0 + A = A$

(d) $A + (-A) = -A + A = 0$

(e) $\lambda(A + B) = \lambda A + \lambda B$

(f) $(\lambda + \mu)A = \lambda A + \mu A$

(g) $\lambda(\mu A) = (\lambda \mu)A$

(h) $1A = A$
Matrix Multiplication

Definition

Let $A$ be $n \times m$ and $B$ be $m \times p$. The matrix product $C = AB$ is the $n \times p$ matrix with entries

$$c_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$$
Special Matrices

Definition

- A **square** matrix has $m = n$
- A **diagonal** matrix $D = [d_{ij}]$ is square with $d_{ij} = 0$ when $i \neq j$
- The **identity matrix of order** $n$, $I_n = [\delta_{ij}]$, is diagonal with

$$
\delta_{ij} = \begin{cases} 
1, & \text{if } i = j, \\
0, & \text{if } i \neq j.
\end{cases}
$$

Definition

- An **upper-triangular** $n \times n$ matrix $U = [u_{ij}]$ has

$$
u_{ij} = 0, \quad \text{if } i = j + 1, \ldots, n.
$$

- A **lower-triangular** $n \times n$ matrix $L = [l_{ij}]$ has

$$
l_{ij} = 0, \quad \text{if } i = 1, \ldots, j - 1.
$$
Theorem

Let $A$ be $n \times m$, $B$ be $m \times k$, $C$ be $k \times p$, $D$ be $m \times k$, and $\lambda$ a real number.

(a) $A(BC) = (AB)C$

(b) $A(B + D) = AB + AD$

(c) $I_mB = B$ and $BI_k = B$

(d) $\lambda(AB) = (\lambda A)B = A(\lambda B)$
Matrix Inversion

Definition

- An $n \times n$ matrix $A$ is *nonsingular or invertible* if $n \times n$ $A^{-1}$ exists with $AA^{-1} = A^{-1}A = I$
- The matrix $A^{-1}$ is called the *inverse* of $A$
- A matrix without an inverse is called *singular or noninvertible*

Theorem

For any nonsingular $n \times n$ matrix $A$,

(a) $A^{-1}$ is unique

(b) $A^{-1}$ is nonsingular and $(A^{-1})^{-1} = A$

(c) If $B$ is nonsingular $n \times n$, then $(AB)^{-1} = B^{-1}A^{-1}$
Definition

- The *transpose* of $n \times m \ A = [a_{ij}]$ is $m \times n \ A^t = [a_{ji}]$
- A square matrix $A$ is called *symmetric* if $A = A^t$

Theorem

(a) $(A^t)^t = A$
(b) $(A + B)^t = A^t + B^t$
(c) $(AB)^t = B^t A^t$
(d) *if* $A^{-1}$ *exists, then* $(A^{-1})^t = (A^t)^{-1}$
Determinants

**Definition**

(a) If \( A = [a] \) is a \( 1 \times 1 \) matrix, then \( \det A = a \)

(b) If \( A \) is \( n \times n \), the minor \( M_{ij} \) is the determinant of the \((n-1) \times (n-1)\) submatrix deleting row \( i \) and column \( j \) of \( A \)

(c) The cofactor \( A_{ij} \) associated with \( M_{ij} \) is \( A_{ij} = (-1)^{i+j} M_{ij} \)

(d) The determinant of \( n \times n \) matrix \( A \) for \( n > 1 \) is

\[
\det A = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}
\]

or

\[
\det A = \sum_{i=1}^{n} a_{ij} A_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}
\]
Properties

**Theorem**

(a) If any row or column of $A$ has all zeros, then $\det A = 0$

(b) If $A$ has two rows or two columns equal, then $\det A = 0$

(c) If $\tilde{A}$ comes from $(E_i) \leftrightarrow (E_j)$ on $A$, then $\det \tilde{A} = -\det A$

(d) If $\tilde{A}$ comes from $(\lambda E_i) \leftrightarrow (E_i)$ on $A$, then $\det \tilde{A} = \lambda \det A$

(e) If $\tilde{A}$ comes from $(E_i + \lambda E_j) \leftrightarrow (E_i)$ on $A$, with $i \neq j$, then $\det \tilde{A} = \det A$

(f) If $B$ is also $n \times n$, then $\det AB = \det A \det B$

(g) $\det A^t = \det A$

(h) When $A^{-1}$ exists, $\det A^{-1} = (\det A)^{-1}$

(i) If $A$ is upper/lower triangular or diagonal, then $\det A = \prod_{i=1}^{n} a_{ii}$
Theorem

The following statements are equivalent for any $n \times n$ matrix $A$:

(a) The equation $Ax = 0$ has the unique solution $x = 0$
(b) The system $Ax = b$ has a unique solution for any $b$
(c) The matrix $A$ is nonsingular; that is, $A^{-1}$ exists
(d) $\det A \neq 0$
(e) Gaussian elimination with row interchanges can be performed on the system $Ax = b$ for any $b$
The \textit{k-th Gaussian transformation matrix} is defined by

\[
M^{(k)} = \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & & & & \\
\vdots & & \ddots & & & \\
\vdots & & & \ddots & & \\
\vdots & & & & \ddots & \\
0 & \cdots & \cdots & \cdots & -m_{k+1,k} & \ddots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
\vdots & \vdots & \vdots & & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & 0 & \ddots \\
0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 1
\end{bmatrix}
\]
Gaussian elimination can be written as

\[ A^{(n)} = M^{(n-1)} \cdots M^{(1)} A = \begin{bmatrix}
    a^{(1)}_{11} & a^{(1)}_{12} & \cdots & a^{(1)}_{1n} \\
    0 & a^{(2)}_{22} & \cdots & \vdots \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & \cdots & 0 & a^{(n-1)}_{n-1,n} \\
    0 & \cdots & 0 & a_{nn}
\end{bmatrix} \]
Reversing the elimination steps gives the inverses:

\[ L^{(k)} = [M^{(k)}]^{-1} = \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & & & \\
\vdots & \ddots & \ddots & \ddots & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \\
\vdots & \ddots & 0 & \ddots & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \\
0 & \cdots & 0 & m_{n,k} & 0 & \cdots & 0 \\
\end{bmatrix} \]

and we have

\[
LU = L^{(1)} \cdots L^{(n-1)} \cdots M^{(n-1)} \cdots M^{(1)} A
= [M^{(1)}]^{-1} \cdots [M^{(n-1)}]^{-1} \cdots M^{(n-1)} \cdots M^{(1)} A = A
\]
**Theorem**

If Gaussian elimination can be performed on the linear system $Ax = b$ without row interchanges, $A$ can be factored into the product of lower-triangular $L$ and upper-triangular $U$ as $A = LU$, where $m_{ji} = \frac{a_{ji}^{(i)}}{a_{ii}^{(i)}}$:

$$
U = \begin{bmatrix}
    a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\
    0 & a_{22}^{(2)} & \ddots & \vdots \\
    \vdots & \ddots & \ddots & a_{n-1,n}^{(n-1)} \\
    0 & \cdots & 0 & a_{nn}^{(n)}
\end{bmatrix},
\quad
L = \begin{bmatrix}
    1 & 0 & \cdots & 0 \\
    m_{21} & 1 & \ddots & \vdots \\
    \vdots & \ddots & \ddots & 0 \\
    m_{n1} & \cdots & m_{n,n-1} & 1
\end{bmatrix}
$$
Suppose \( k_1, \ldots, k_n \) is a permutation of 1, \ldots, n. The permutation matrix \( P = (p_{ij}) \) is defined by

\[
p_{ij} = \begin{cases} 
1, & \text{if } j = k_i, \\
0, & \text{otherwise}.
\end{cases}
\]

(i) \( PA \) permutes the rows of \( A \):

\[
PA = \begin{bmatrix} 
a_{k_11} & \cdots & a_{k_1n} \\
\vdots & \ddots & \vdots \\
a_{k_n1} & \cdots & a_{k_nn}
\end{bmatrix}
\]

(ii) \( P^{-1} \) exists and \( P^{-1} = P^t \)

Gaussian elimination with row interchanges then becomes:

\[
A = P^{-1}LU = (P^tL)U
\]
Diagonally Dominant Matrices

Definition

The $n \times n$ matrix $A$ is said to be strictly diagonally dominant when

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

Theorem

A strictly diagonally dominant matrix $A$ is nonsingular, Gaussian elimination can be performed on $Ax = b$ without row interchanges, and the computations will be stable.
Positive Definite Matrices

**Definition**

A matrix $A$ is *positive definite* if it is symmetric and if $\mathbf{x}^t A \mathbf{x} > 0$ for every $\mathbf{x} \neq \mathbf{0}$.

**Theorem**

If $A$ is an $n \times n$ positive definite matrix, then

(a) $A$ has an inverse

(b) $a_{ii} > 0$

(c) $\max_{1 \leq k, j \leq n} |a_{kj}| \leq \max_{1 \leq i \leq n} |a_{ii}|$

(d) $(a_{ij})^2 < a_{ii} a_{jj}$ for $i \neq j$
Definition

A leading principal submatrix of a matrix $A$ is a matrix of the form

$$A_k = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix}$$

for some $1 \leq k \leq n$.

Theorem

A symmetric matrix $A$ is positive definite if and only if each of its leading principal submatrices has a positive determinant.
Theorem

The symmetric matrix $A$ is positive definite if and only if Gaussian elimination without row interchanges can be done on $Ax = b$ with all pivot elements positive, and the computations are then stable.

Corollary

The matrix $A$ is positive definite if and only if it can be factored $A = LDL^t$ where $L$ is lower triangular with 1’s on its diagonal and $D$ is diagonal with positive diagonal entries.

Corollary

The matrix $A$ is positive definite if and only if it can be factored $A = LL^t$, where $L$ is lower triangular with nonzero diagonal entries.
Definition

An $n \times n$ matrix is called a band matrix if $p, q$ exist with $1 < p, q < n$ and $a_{ij} = 0$ when $p \leq j - i$ or $q \leq i - j$. The bandwidth is $w = p + q - 1$.

A tridiagonal matrix has $p = q = 2$ and bandwidth 3.

Theorem

Suppose $A = [a_{ij}]$ is tridiagonal with $a_{i,i-1}a_{i,i+1} \neq 0$. If $|a_{11}| > |a_{12}|$, $|a_{ii}| \geq |a_{i,i-1}| + |a_{i,i+1}|$, and $|a_{nn}| > |a_{n,n-1}|$, then $A$ is nonsingular.