UC Berkeley Math 228B, Spring 2024: Problem Set 4

Prof. Per-Olof Persson (persson@berkeley.edu)

Due March 15

1. Consider the boundary value problem

$$u''''(x) = f(x) \equiv 480x - 120, \qquad \text{for } x \in (0,1) \tag{1}$$
$$u(0) = u'(0) = u(1) = u'(1) = 0 \tag{2}$$

   (a) Derive the following Galerkin formulation for the problem (1)-(2) on some appropriate function space V_h: Find $u_h \in V_h$ such that

$$\int_0^1 u_h''(x)v''(x)\,dx = \int_0^1 f(x)v(x)\,dx, \qquad \forall v \in V_h. \tag{3}$$

   (b) Define the triangulation $T_h = \{K_1, K_2\}$, where $K_1 = [0, \frac{1}{2}]$ and $K_2 = [\frac{1}{2}, 1]$, and the function space

$$V_h = \{v \in C^1([0,1]) : v\big|_K \in \mathbb{P}_3(K)\ \forall K \in T_h,\ v(0) = v'(0) = v(1) = v'(1) = 0\}. \tag{4}$$

   Find a basis $\{\varphi_i\}$ for V_h. *Hint*: Consider Hermite polynomials on each element.

   (c) Solve the Galerkin problem (3) using your basis functions. Plot the numerical solution $u_h(x)$ and the true solution $u(x)$.

2. Implement a Julia function with the syntax

```
u = fempoi(p,t,e)
```

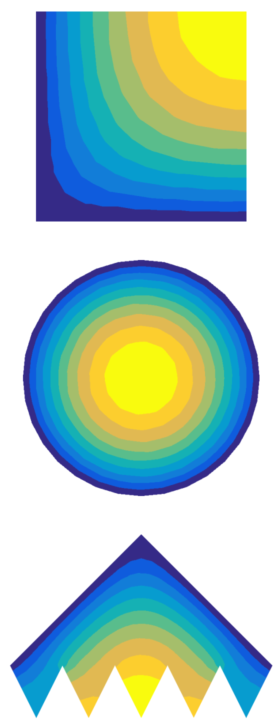
that solves Poissons's equation $-\nabla^2 u(x,y) = 1$ on the domain described by the unstructured triangular mesh `p,t`. The boundary conditions are homogeneous Neumann ($n \cdot \nabla u = 0$) except for the nodes in the array `e` which are homogeneous Dirichlet ($u = 0$).

Here are a few examples for testing the function:

```julia
# Square, Dirichlet left/bottom
pv = Float64[0 0; 1 0; 1 1; 0 1; 0 0]
p, t, e = pmesh(pv, 0.15, 0)
e = e[@. (p[e,1] < 1e-6) | (p[e,2] < 1e-6)]
u = fempoi(p, t, e)
tplot(p, t, u)

# Circle, all Dirichlet
n = 32; phi = 2pi*(0:n)/n
pv = [cos.(phi)  sin.(phi)]
p, t, e = pmesh(pv, 2pi/n, 0)
u = fempoi(p, t, e)
tplot(p, t, u)

# Generic polygon geometry, mixed Dirichlet/Neumann
x = 0:.1:1
y = 0.1*(-1).^(0:10)
pv = [x y; .5 .6; 0 .1]
p, t, e = pmesh(pv, 0.04, 0)
e = e[@. p[e,2] > (.6 - abs(p[e,1] - 0.5) - 1e-6)]
u = fempoi(p, t, e)
tplot(p, t, u)
```



3. Implement a Julia function with the syntax

```
errors = poiconv(pv, hmax, nrefmax)
```

that solves the all-Dirichlet Poisson problem for the polygon `pv`, using the mesh parameters `hmax` and `nref = 0,1,...,nrefmax`. Consider the solution on the finest mesh the exact solution, and compute the max-norm of the errors at the nodes for all the other solutions (note that this is easy given how the meshes were refined – the common nodes appear first in each mesh). The output `errors` is a vector of length `nrefmax` containing all the errors.

Test the function using the commands below, which makes a convergence plot and estimates the rates:

```julia
hmax = 0.15
pv_square = Float64[0 0; 1 0; 1 1; 0 1; 0 0]
pv_polygon = Float64[0 0; 1 0; .5 .5; 1 1; 0 1; 0 0]

errors_square = poiconv(pv_square, hmax, 3)
errors_polygon = poiconv(pv_polygon, hmax, 3)
errors = [errors_square errors_polygon]

clf()
loglog(hmax ./ [1,2,4], errors)
rates = @. log2(errors[end-1,:]) - log2(errors[end,:])
```

Code Submission: Your Julia file needs to define the functions `fempoi` and `poiconv`, with exactly the requested names and input/output arguments, as well as any other supporting functions and variables that are required for your functions to run correctly.

