Implementation of Finite Element Methods

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Math 228B Numerical Solutions of Differential Equations

The Poisson Problem in 2-D

- Consider the problem

$$-\nabla^2 u = f \text{ in } \Omega$$
$$n \cdot \nabla u = g \text{ on } \Gamma$$

  for a domain Ω with boundary Γ
- Seek solution $\hat{u} \in \hat{X}$, multiply by a test function $v \in \hat{X}$, and integrate:

$$\int_\Omega -\nabla^2 \hat{u} v \, d\Omega = \int_\Omega f v \, d\Omega$$

- Apply the divergence theorem and use the Neumann condition, to get the Galerkin form

$$\int_\Omega \nabla \hat{u} \nabla v \, d\Omega = \int_\Omega f v \, d\Omega + \oint_\Gamma g v \, ds$$

Finite Element Formulation

- Expand in basis $\hat{u} = \sum_i \hat{u}_i \phi_i(x)$, insert into the Galerkin form, and set $v = \phi_i$, $i = 1, \ldots, n$:

$$\int_\Omega \left[\sum_{j=1}^{n} \hat{u}_j \nabla \phi_j \right] \cdot \nabla \phi_i \, d\Omega = \int_\Omega f \phi_i \, d\Omega + \oint_\Gamma g \phi_i \, ds$$

Switch order of integration and summation to get the finite element formulation:

$$\sum_{j=1}^{n} A_{ij} \hat{u}_j = b_i, \quad \text{or} \quad A\hat{u} = b$$

where

$$A_{ij} = \int_\Omega \nabla \phi_i \cdot \nabla \phi_j \, d\Omega, \quad b_i = \int_\Omega f \phi_i + \oint_\Gamma g \phi_i \, ds$$

Discretization

- Find a tringulation of the domain Ω into triangular elements T^k, $k = 1, \ldots, K$ and nodes x_i, $i = 1, \ldots, n$
- Consider the space $\hat{X}$ of continuous functions that are linear within each element
- Use a nodal basis $\hat{X} = \text{span}\{\phi_1, \ldots, \phi_n\}$ defined by

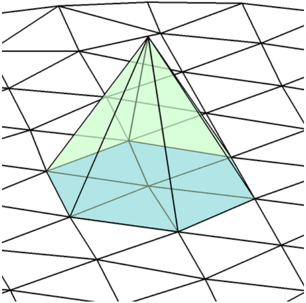
$$\phi_i \in \hat{X}, \quad \phi_i(x_j) = \delta_{ij}, \quad 1 \leq i, j \leq n$$

- A function $v \in \hat{X}$ can then be written

$$v = \sum_{i=1}^{n} v_i \phi_i(x)$$

with the nodal interpretation

$$v(x_j) = \sum_{i=1}^{n} v_i \phi_i(x) = \sum_{i=1}^{n} v_i \delta_{ij} = v_j$$



Local Basis Functions

- Consider a triangular element T^k with local nodes x_1^k, x_2^k, x_3^k
- The local basis functions $\mathcal{H}_1^k, \mathcal{H}_2^k, \mathcal{H}_3^k$ are linear functions:

$$\mathcal{H}_\alpha^k = c_\alpha^k + c_{x,\alpha}^k x + c_{y,\alpha}^k y, \quad \alpha = 1, 2, 3$$

  with the property that $\mathcal{H}_\alpha^k(x_\beta) = \delta_{\alpha\beta}$, $\beta = 1, 2, 3$
- This leads to linear systems of equations for the coefficients:

$$\begin{pmatrix} 1 & x_1^k & y_1^k \\ 1 & x_2^k & y_2^k \\ 1 & x_3^k & y_3^k \end{pmatrix} \begin{pmatrix} c_\alpha^k \\ c_{x,\alpha}^k \\ x_{y,\alpha}^k \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

  or $C = V^{-1}$ with coefficient matrix C and Vandermonde matrix V

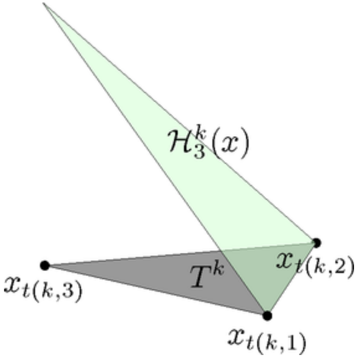
Elementary Matrices and Loads

- The elementary matrix for an element T^k becomes

$$A^k_{\alpha\beta} = \int_{T^k} \frac{\partial \mathcal{H}^k_\alpha}{\partial x}\frac{\partial \mathcal{H}^k_\beta}{\partial x} + \frac{\partial \mathcal{H}^k_\alpha}{\partial y}\frac{\partial \mathcal{H}^k_\beta}{\partial y}\, d\Omega$$
$$= \mathrm{Area}^k (c^k_{x,\alpha}c^k_{x,\beta} + c^k_{y,\alpha}c^k_{y,\beta}), \quad \alpha, \beta = 1, 2, 3$$

- The elementary load becomes

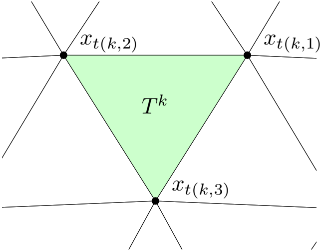
$$b^k_\alpha = \int_{T^k} f\,\mathcal{H}^k_\alpha\, d\Omega$$
$$= (\text{if } f \text{ constant})$$
$$= \frac{\mathrm{Area}^k}{3} f, \quad \alpha = 1, 2, 3$$



Assembly, The Stamping Method

- Assume a local-to-global mapping $t(k, \alpha)$, giving the global node number for local node number α in element k
- The global linear system is then obtained from the elementary matrices and loads by the stamping method:

$A = 0$, $b = 0$
for $k = 1, \ldots, K$
$$A(t(k,:), t(k,:)) = A(t(k,:), t(k,:)) + A^k$$
$$b(t(k,:)) = b(t(k,:)) + b^k$$



Dirichlet Conditions

- Suppose Dirichlet conditions $u = u_D$ are imposed on part of the boundary Γ_D
- Enforce $\hat{u}_i = u_D$ for all nodes i on Γ_D directly in the linear system of equations:

$$
\begin{array}{c c}
 & \begin{array}{ccccccc} & & & i & & & \end{array} \\
i & \left(\begin{array}{ccccccc} & & & & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & & & & & \end{array}\right)
\end{array}
\hat{u} = \left(\begin{array}{c} \\ u_D \\ \\ \end{array}\right)
$$