Mesh Generation

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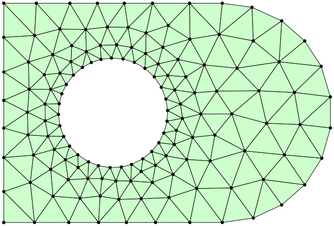
Math 228B Numerical Solutions of Differential Equations

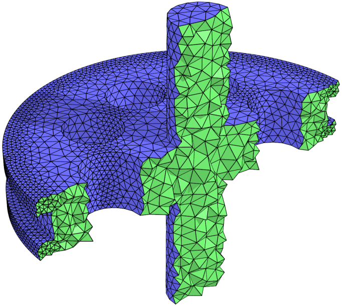
Mesh Generation

Motivation: Most numerical methods for PDEs require a mesh for non-trivial domains

Various methods might use different components of the mesh:

- Nodes (vertices)
- Edges (faces in 3D)
- Elements





Structured vs. Unstructured Meshes

Natural classification of meshes based on connectivity of nodes:

- In *structured* meshes, all nodes have the same connections to their neighbors (at least away from the boundaries)
- *Unstructured* meshes allow for arbitrary connectivities (as long as the mesh remains conforming)
- *Hybrid* meshes combine the two, e.g. by having structured parts in certain areas of the domain

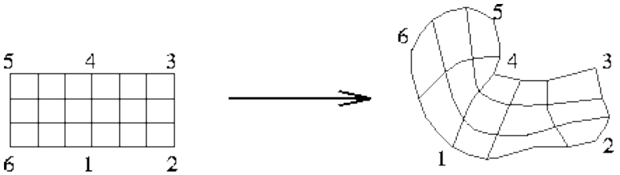
Structured Mesh Generation

Why Structured Meshes?

- Lead to very efficient numerical methods
- High quality for sufficiently simple geometries
- Larger grid control when high anisotropy is required
- Multi-block approach allows for realistic geometries

Single-Block Grid Generation

- Construct a one-to-one mapping between a rectangular computational domain and a physical domain
- Ideally, grid size in physical space should be dictated by solver/solution requirements
- Ensure grid quality e.g. smoothness, orthogonality



Single Block Grid Generation - Creating the Mapping

- Transfinite Interpolation (TFI)
- Conformal Mapping
- Solving PDE's
  - Elliptic
  - Parabolic/Hyperbolic

Algebraic Mappings

- Construct a mapping between the boundaries of the unit square (cube) and the boundaries of an "arbitrary" region which is topologically equivalent
- Combine 1D interpolants using Boolean sums to construct mapping - Transfinite Interpolation (TFI)
- Not guaranteed to be one-to-one
- Orthogonality not guaranteed
- Very Fast
- Quite General
- **Grid quality not always assured**

Algebraic Mappings - 1D Interpolants

- General 1D interpolant of $f(x)$ for $x \in (0,1)$

$$\hat{f}(x) \equiv \Pi_x f = \sum_{i=0}^{L} \sum_{n=0}^{P} \alpha_i^n(x) \left. \frac{d^n f}{dx^n} \right|_{x=x_i}$$

  - $\alpha_i^n(x)$ are the blending functions
- Examples
  - Linear Lagrange interpolation - $P = 0, L = 1$

$$\Pi_x f = (1-x) f(0) + x f(1)$$

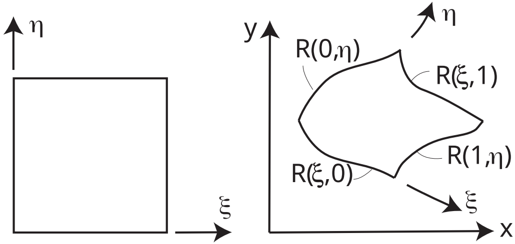
  - Quadratic Lagrange interpolation - $P = 0, L = 2$

$$\Pi_x f = (2x^2 - 3x + 1) f(0) + (4x - 4x^2) f(0.5) + (2x^2 - x) f(1)$$

  - Hermite interpolation - $P = 1, L = 1$

$$\begin{aligned} \Pi_x f &= (2x^3 - 3x^2 + 1) f(0) + (3x^2 - 2x^3) f(1) + \\ &\quad (x^3 - 2x^2 + x) f'(0) + (x^3 - x^2) f'(1) \end{aligned}$$

Algebraic Mappings - Transfinite Interpolation



- Start from 1D boundary mappings of $\mathbf{R} \equiv (x, y)$, e.g.
$$\mathbf{R}(\xi, 0), \mathbf{R}(\xi, 1), \mathbf{R}(0, \eta), \mathbf{R}(1, \eta)$$
- Construct 1D interpolants in the ξ and η directions (e.g. linear)

$$\begin{aligned}
\Pi_\xi \mathbf{R} &= (1-\xi)\mathbf{R}(0,\eta) + \xi \mathbf{R}(1,\eta) \\
\Pi_\eta \mathbf{R} &= (1-\eta)\mathbf{R}(\xi,0) + \eta \mathbf{R}(\xi,1)
\end{aligned}$$

Algebraic Mappings - Transfinite Interpolation

- Construct two-dimensional interpolant by doing the Boolean sum

$$\hat{\mathbf{R}}(\xi,\eta) = (\Pi_\xi \oplus \Pi_\eta)\mathbf{R} = (\Pi_\xi + \Pi_\eta - \Pi_\xi \Pi_\eta)\mathbf{R}$$

Expanding:

$$\hat{\mathbf{R}}(\xi,\eta) = (1-\xi,\xi)\begin{pmatrix} \mathbf{R}(0,\eta) \\ \mathbf{R}(1,\eta) \end{pmatrix} + (\mathbf{R}(\xi,0),\mathbf{R}(\xi,1))\begin{pmatrix} 1-\eta \\ \eta \end{pmatrix}$$

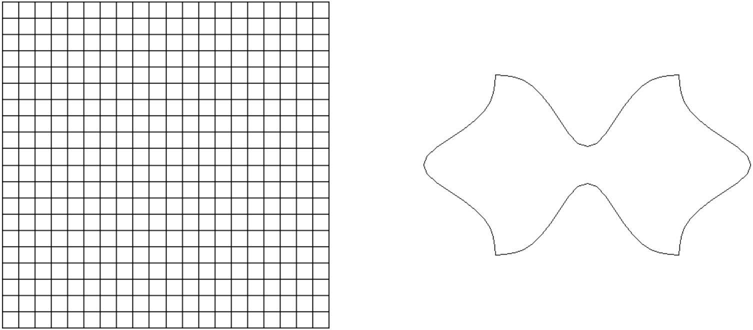
$$-(1-\xi,\xi)\begin{pmatrix} \mathbf{R}(0,0) & \mathbf{R}(0,1) \\ \mathbf{R}(1,0) & \mathbf{R}(1,1) \end{pmatrix}\begin{pmatrix} 1-\eta \\ \eta \end{pmatrix}$$

$$= (1-\xi)\mathbf{R}(0,\eta) + \xi\mathbf{R}(1,\eta) + (1-\eta)\mathbf{R}(\xi,0) + \eta\mathbf{R}(\xi,1)$$

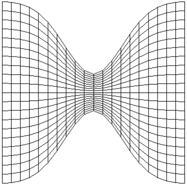
$$-(1-\xi)(1-\eta)\mathbf{R}(0,0) - (1-\xi)\eta\mathbf{R}(0,1) - \xi(1-\eta)\mathbf{R}(1,0) - \xi\eta\mathbf{R}(1,1)$$

- **Important property**: Preserves $\mathbf{R}$ at the domain boundary
- Extends to **general 1D interpolants** and **any dimension**

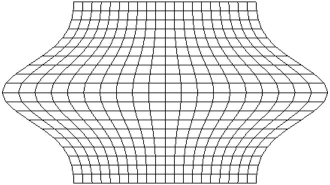
Algebraic Mappings - Example



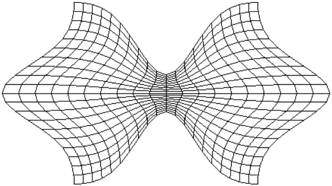
Algebraic Mappings - Example



$\Pi_\xi \mathbf{R}$



$\Pi_\eta \mathbf{R}$



$(\Pi_\xi \oplus \Pi_\eta)\mathbf{R}$

Algebraic Mappings - Grid Control

- Use non-regular subdivisions in (ξ, η) (e.g. exponential functions) to obtain desired element sizes in (x, y)
- Use derivative boundary conditions to enforce boundary orthogonality

$$\frac{\partial \mathbf{R}}{\partial \xi} \cdot \frac{\partial \mathbf{R}}{\partial \eta} = 0$$

Conformal Mapping

- An analytic function $\alpha = f(z)$ such that $\dfrac{df}{dz} \neq 0$ defines a one-to-one (conformal) mapping between $z = x + iy$ and $\alpha = \xi + i\eta$, or between (x, y) and (ξ, η).
- The functions $\xi(x, y)$ and $\eta(x, y)$ satisfy the Cauchy- Riemann equations (e.g. $\xi_x = \eta_y$, and $\eta_x = -\xi_y$) and as a consequence, they are harmonic

$$\nabla^2 \xi = 0, \qquad \nabla^2 \eta = 0 \qquad \text{(smoothness)}$$

- Preserve angles (grid orthogonality)
- Preserve ratios
- Lead to high quality grids
- **Limited to 2D**

Conformal Mapping Transformations

- Joukowski (maps circle of radius c to segment $[-2c, 2c]$)

$$\alpha = z + \frac{c^2}{z}, \quad \text{or} \qquad \frac{\alpha + 2c}{\alpha - 2c} = \left(\frac{z+c}{z-c}\right)^2$$

- Karman-Trefftz

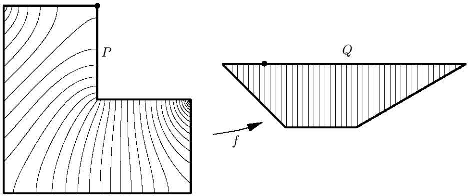
$$\frac{\alpha + 2c}{\alpha - 2c} = \left(\frac{z+c}{z-c}\right)^n$$

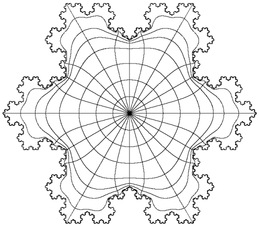
- Schwarz-Christoffel (maps polygon into half plane)

$$\frac{d\alpha}{dz} = K \prod_{k=1}^{n} \left(1 - \frac{z}{z_k}\right)^{\beta_k}$$

Conformal Mapping - Schwarz-Christoffel

Ref. "Schwarz-Christofell Mapping", *Driscoll and Trefethen*, Cambridge Univeristy Press, 2002.





PDE Grid Generation

- Construct mapping by solving a PDE
  - Elliptic Equations (smooth grids)

$$\nabla^2 \xi(x, y) = P(x, y), \quad \nabla^2 \eta(x, y) = Q(x, y)$$

  - Hyperbolic equations (orthogonal grids)

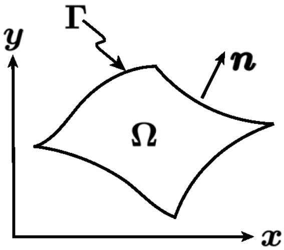
$$\begin{aligned} x_\xi y_\eta - x_\eta y_\xi &= J && \text{(size control)} \\ x_\xi x_\eta + y_\xi y_\eta &= 0 && \text{(orthogonality)} \end{aligned}$$

- Most widely used approach
- **Grids usually have high quality**

We are interested in solving

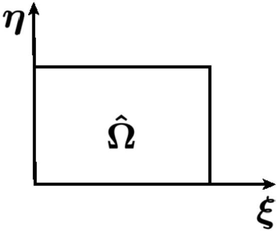
$$\begin{aligned} -\nabla^2 \xi &= P && \text{in} \quad \Omega \\ \xi &= g && \text{on} \quad \Gamma_D \\ \frac{\partial \xi}{\partial n} &= h && \text{on} \quad \Gamma_N = \Gamma \backslash \Gamma_D \end{aligned}$$

where P, g, and h are given.

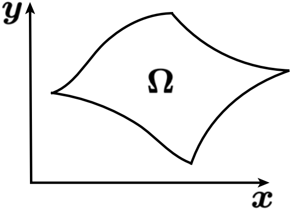


Similarly for $\eta(x, y)$

Elliptic Grid Generation



$$x = x(\xi, \eta)$$
$$y = y(\xi, \eta)$$
$$\longrightarrow$$



? ?

$$\nabla^2 \xi = P$$

Can we determine an equivalent problem to be solved on $\hat{\Omega}$?

Elliptic Grid Generation

$$\begin{aligned} \xi &= \xi(x, y) \\ \eta &= \eta(x, y) \end{aligned} \qquad\qquad \begin{aligned} x &= x(\xi, \eta) \\ y &= y(\xi, \eta) \end{aligned}$$

$$\begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} \qquad \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix} \begin{pmatrix} d\xi \\ d\eta \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} = \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix}^{-1} = \frac{1}{J} \begin{pmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{pmatrix}$$

$$J = x_\xi y_\eta - x_\eta y_\xi$$

Elliptic Grid Generation

$$\xi_x = \frac{y_\eta}{J} \qquad \xi_y = -\frac{x_\eta}{J}$$

$$\eta_x = -\frac{y_\xi}{J} \qquad \eta_y = \frac{x_\xi}{J}$$

and

$$\begin{aligned}
\xi_{xx} &= \frac{\partial}{\partial x}(\xi_x) = \left(\xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}\right)\left(\frac{y_\eta}{J}\right) \\
&= \frac{1}{J}\left(y_\eta \frac{\partial}{\partial \xi} - y_\xi \frac{\partial}{\partial \eta}\right)\left(\frac{y_\eta}{J}\right) \\
&= \dots \\
\xi_{yy} &= \dots
\end{aligned}$$

Elliptic Grid Generation - Thompson's Equations

Finally, $\xi_{xx} + \xi_{yy} = 0$ and $\eta_{xx} + \eta_{yy} = 0$, become

$$\boxed{\begin{array}{rcl} a x_{\xi\xi} - 2b x_{\xi\eta} + c x_{\eta\eta} & = & 0 \\ a y_{\xi\xi} - 2b y_{\xi\eta} + c y_{\eta\eta} & = & 0 \end{array}}$$

a, b, c depend on the mapping.

$$a = x_\eta^2 + y_\eta^2 \quad b = x_\xi x_\eta + y_\xi y_\eta \quad c = x_\xi^2 + y_\xi^2$$

- These equations can be solved using **central finite differences** on a regular grid in the (ξ, η) domain to determine the (x, y) coordinates of each grid point.
- Picard iteration: Start from initial grid coordinates x, y. Compute a, b, c, solve the PDE, and repeat until convergence.

Elliptic Grid Generation - Grid Control

- Modify grid by e.g. adding source terms to the PDE:

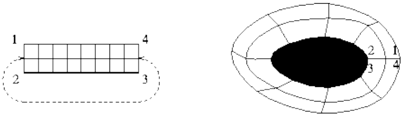
  $\xi_{xx} + \xi_{yy} = P(x, y)$ and $\eta_{xx} + \eta_{yy} = Q(x, y)$

$$\boxed{\begin{array}{rcl} a x_{\xi\xi} - 2b x_{\xi\eta} + c x_{\eta\eta} & = & -J^2(x_\xi P + x_\eta Q) \\ a y_{\xi\xi} - 2b y_{\xi\eta} + c y_{\eta\eta} & = & -J^2(y_\xi P + y_\eta Q) \end{array}}$$

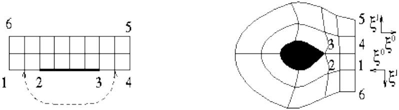
- The functions $P(\xi, \eta)$ and $Q(\xi, \eta)$ can be used to obtain grid control
- Derivative boundary conditons can be used to enforce grid orthogonality at the boundary

Ref: "Numerical Generation of Two-Dimensional Grids by Use of Poisson Equations with Grid Control", *Sorenson and Steger*, in Numerical Grid Generation Techniques, Smith, R.E. (Ed.), NASA-CP-2166, pp. 449-461, 1980

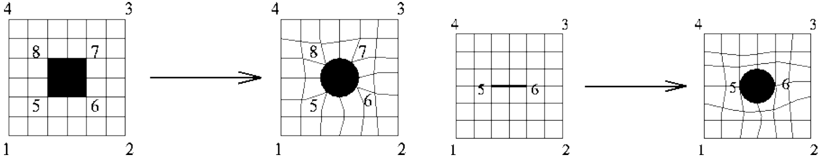
Single-Block Grid Common Topologies



O-Grid



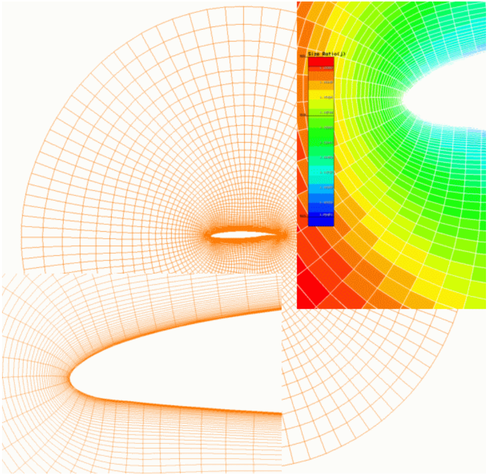
C-Grid

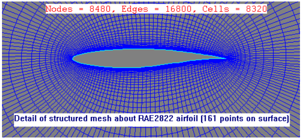


H-Grids

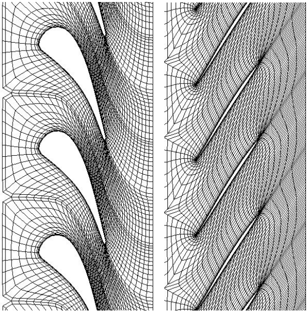
. . . plus combinations

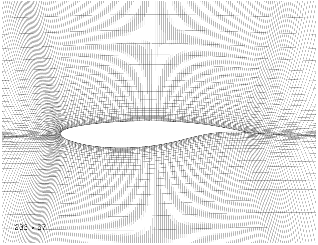
Examples: Single-Block O-Grids

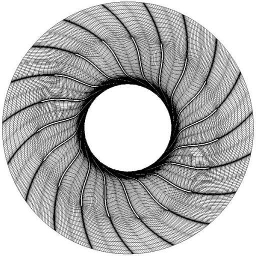


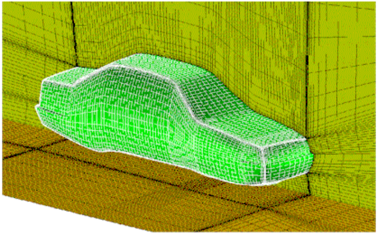


Examples: Single-Block C,H-Grids

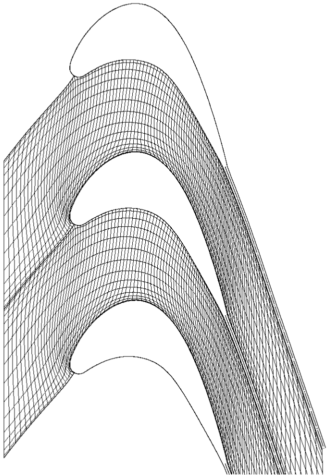




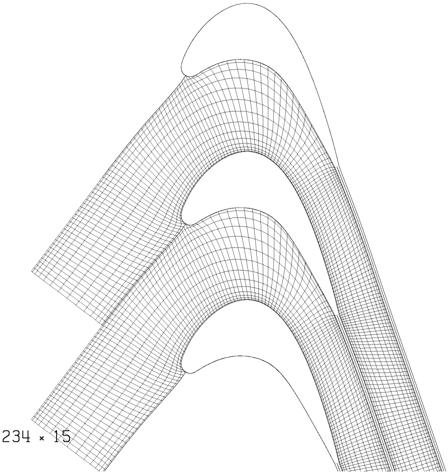




Examples: H-Grids



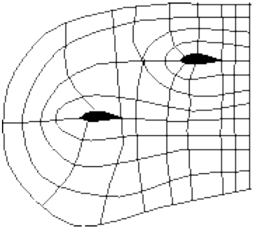
H-Grid



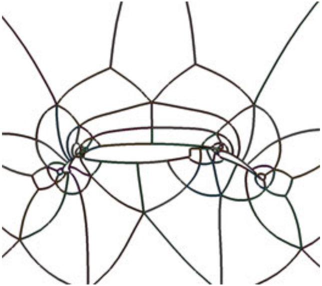
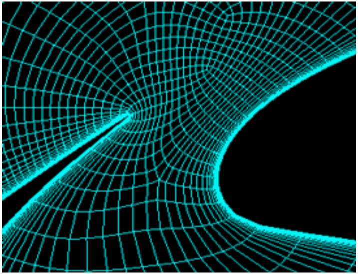
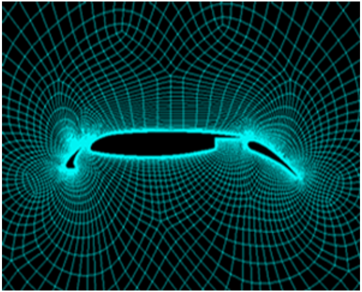
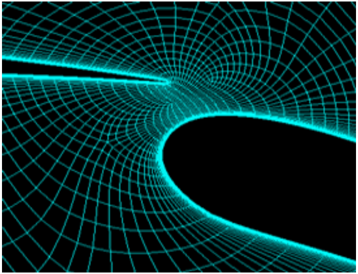
H-Grid/I-Grid

Multi-Block Grid Generation

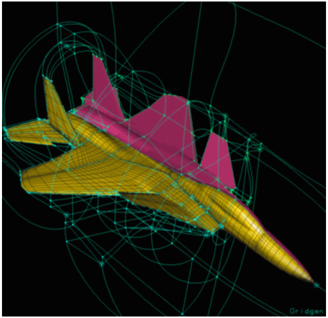
- Subdivide domain into an unstructured assembly of quadrilaterals/hexahedra
- Obtaining block topology automatically is hard
- Obtaining block geometry automatically (e.g. point coordinates) once topology is known is tractable

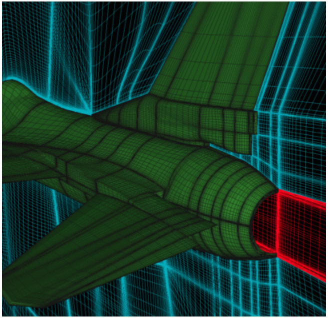


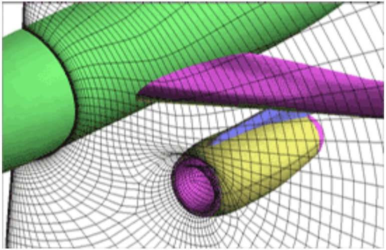
Examples: Multi-Block Grids

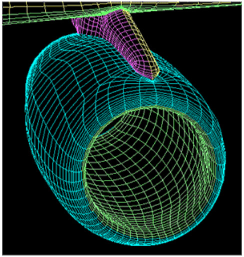





Examples: Multi-Block Grids



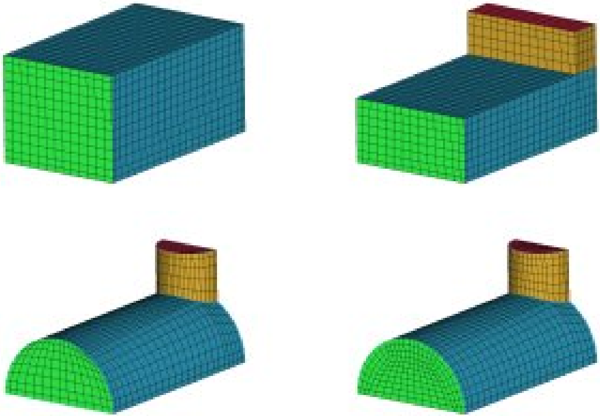






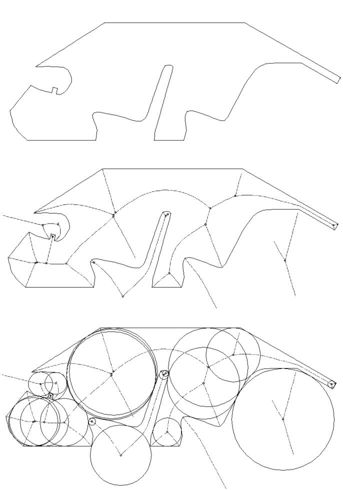
Block Topology Generators

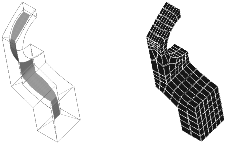
(from ICEM CFD)

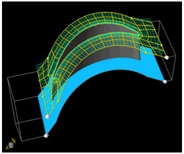


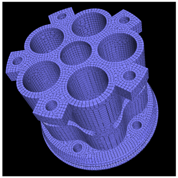
Automatic $H \Rightarrow O$ conversion

Block Topology Generators - Medial Axis Transform (MAT)





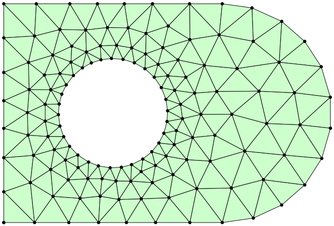


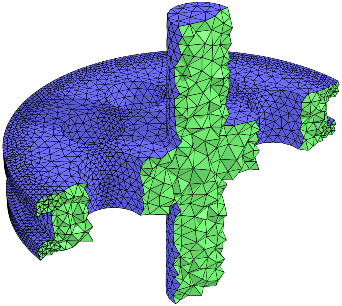


Unstructured Mesh Generation

Unstructured Mesh Generation

- Approximate a domain in $\mathbb{R}^d$ by simple geometric shapes
- Determine node points and element connectivity
- Goal: Resolve the domain accurately with well-shaped elements, but use as few elements as possible
- Applications: Numerical solution of PDEs (FEM, FVM, DGM, BEM), interpolation, computer graphics, visualization

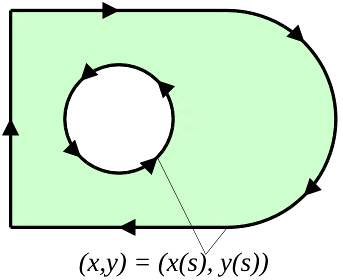




Geometry Representations

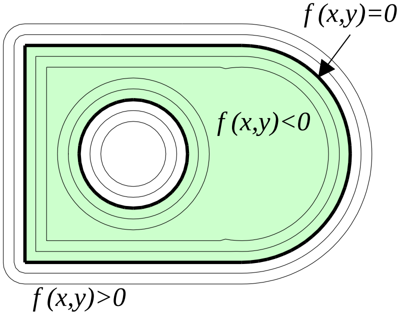
Explicit Geometry

- Parameterized boundaries



Implicit Geometry

- Boundaries from contour



Unstructured Meshing Algorithms

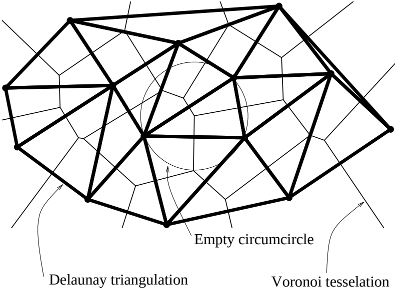
- ***Delaunay refinement***
  - Refine an initial triangulation by inserting centroid points and updating connectivities
  - Efficient and robust, provably good in 2-D
- ***Advancing front***
  - Propagate a layer of elements from boundaries into domain, stitch together at intersection
  - High quality meshes, good for boundary layers, but somewhat unreliable in 3-D

Unstructured Meshing Algorithms

- ***Octree mesh***
  - Create an octree, refine until geometry well resolved, form elements between cell intersections
  - Guaranteed quality even in 3-D, but poor element qualities
- ***DistMesh***
  - Improve initial triangulation by node movements and connectivity updates
  - Easy to understand and use, handles implicit geometries, high element qualities, but non-robust and low performance

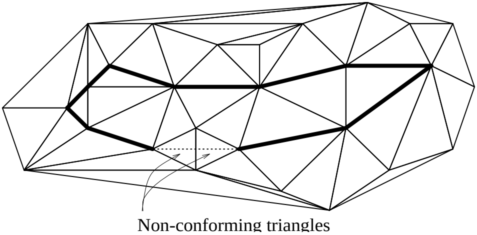
Delaunay Triangulation

- Find non-overlapping triangles that fill the convex hull of a set of points
- Properties:
  - Every edge is shared by at most two triangles
  - The circumcircle of a triangle contains no other input points
  - Maximizes the minimum angle of all the triangles

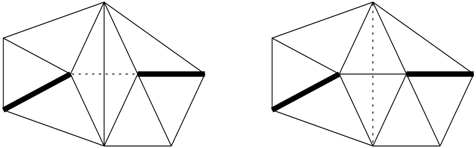


Constrained Delaunay Triangulation

- The Delaunay triangulation might not respect given input edges

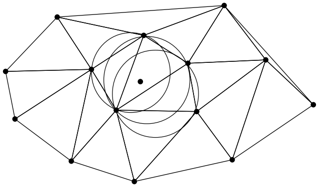


- Use local edge swaps to recover the input edges



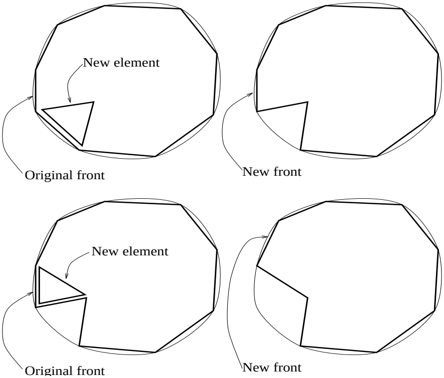
Delaunay Refinement Method

- Algorithm:
  - Form initial triangulation using boundary points and outer box
  - Replace an undesired element (bad or large) by inserting its circumcenter, retriangulate and repeat until mesh is good
- Will converge with high element qualities in 2-D
- Very fast – time almost linear in number of nodes



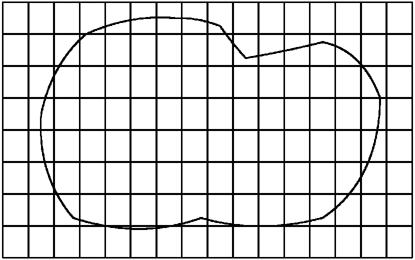
The Advancing Front Method

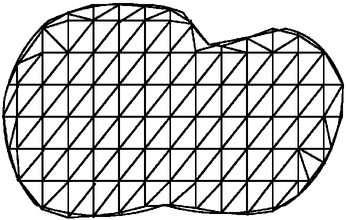
- Discretise the boundary as initial front
- Add elements into the domain and update the front
- When front is empty the mesh is complete



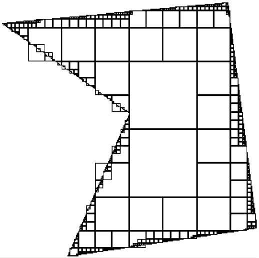
Grid Based and Octree Meshing

- Overlay domain with regular grid, crop and warp edge points to boundary



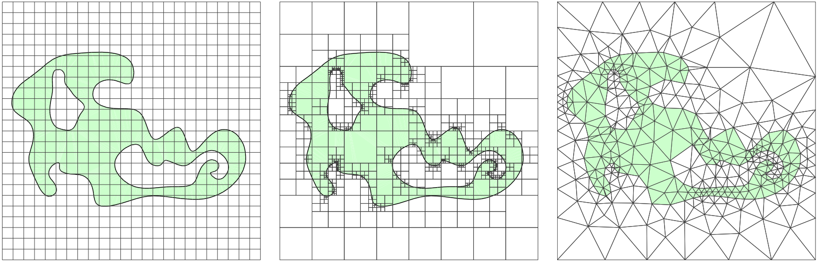


- Octree instead of regular grid gives graded mesh with fewer elements



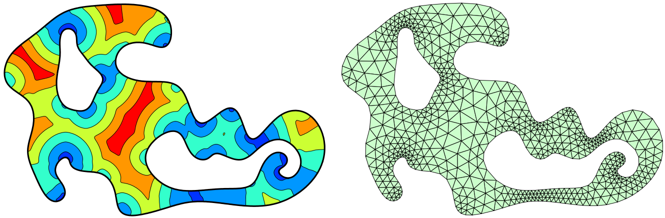
Mesh Size Functions

- Function $h(\boldsymbol{x})$ specifying desired mesh element size
- Many mesh generators need a priori mesh size functions
  - Physically-based methods such as DistMesh
  - Advancing front and Paving methods
- Discretize mesh size function $h(\boldsymbol{x})$ on a *background grid*



Mesh Size Functions

- Based on several factors:
  - Curvature of geometry boundary
  - Local feature size of geometry
  - Numerical error estimates (adaptive solvers)
  - Any user-specified size constraints
- Also: $|\nabla h(\boldsymbol{x})| \leq g$ to limit ratio $G = g + 1$ of neighboring element sizes



Explicit Mesh Size Functions

- A *point-source*

$$h(\boldsymbol{x}) = h_{\mathrm{pnt}} + g|\boldsymbol{x} - \boldsymbol{x}_0|$$

- Any shape, with distance function $\phi(\boldsymbol{x})$

$$h(\boldsymbol{x}) = h_{\mathrm{shape}} + g\phi(\boldsymbol{x})$$

- Combine mesh size functions by $\min$ operator:

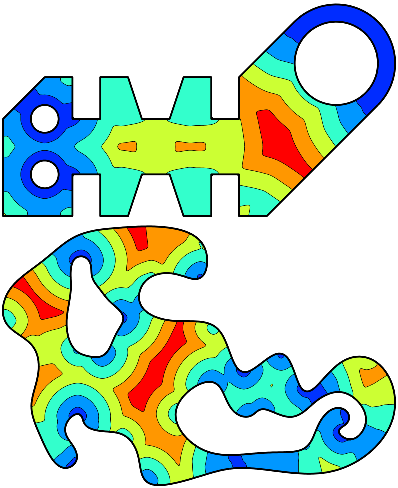
$$h(\boldsymbol{x}) = \min_i h_i(\boldsymbol{x})$$

- For more general $h(\boldsymbol{x})$, solve the *gradient limiting equation* [Persson'05]

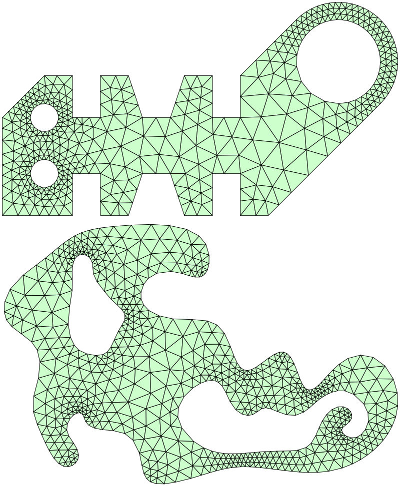
$$\frac{\partial h}{\partial t} + |\nabla h| = \min(|\nabla h|, g),$$
$$h(t = 0, \boldsymbol{x}) = h_0(\boldsymbol{x}).$$

Mesh Size Functions – 2-D Examples

Mesh Size Function $h(\boldsymbol{x})$



Mesh Based on $h(\boldsymbol{x})$

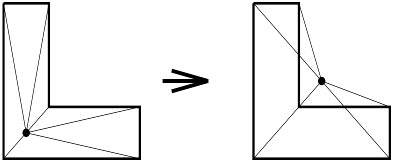


Laplacian Smoothing

- Improve node locations by iteratively moving nodes to average of neighbors:

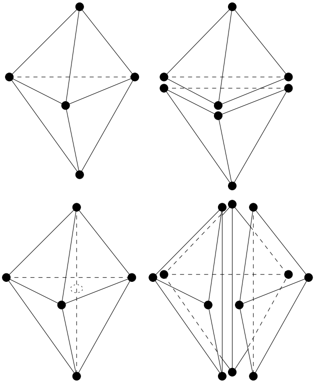
$$\boldsymbol{x}_i \leftarrow \frac{1}{n_i} \sum_{j=1}^{n_i} \boldsymbol{x}_j$$

- Usually a good postprocessing step for Delaunay refinement
- However, element quality can get worse and elements might even invert:



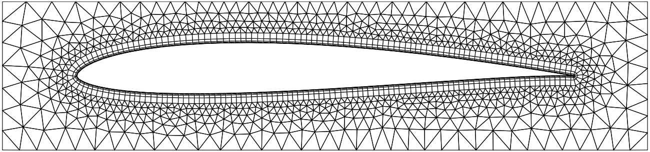
Face and Edge Swapping

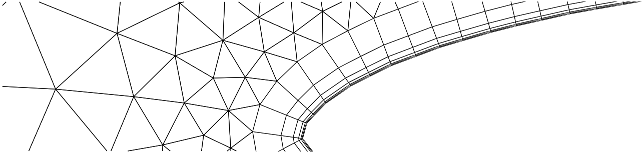
- In 3-D there are several swappings between neighboring elements
- Face and edge swapping important postprocessing of Delaunay meshes



Boundary Layer Meshes

- Unstructured mesh for *offset curve* $\psi(\boldsymbol{x}) - \delta$
- The structured grid is easily created with the distance function





The DistMesh Mesh Generator

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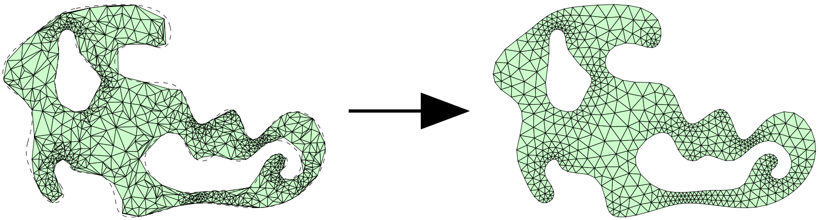
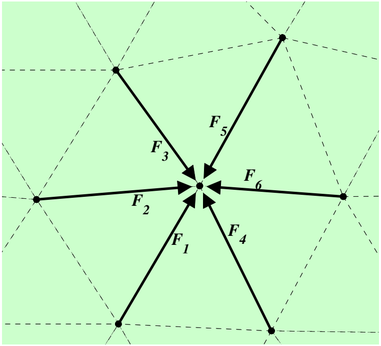
1. Start with *any* topologically correct initial mesh, for example random node distribution and Delaunay triangulation
2. Move nodes to find force equilibrium in edges
   - Project boundary nodes using implicit function ϕ
   - Update element connectivities



Internal Forces



For each *interior* node:

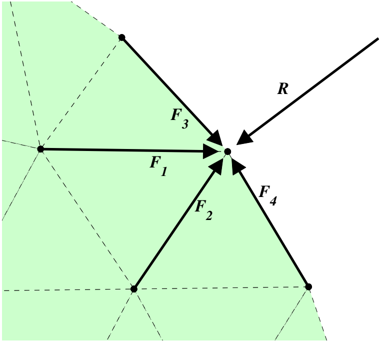
$$\sum_i \boldsymbol{F}_i = 0$$

Repulsive forces depending on edge length ℓ and equilibrium length ℓ_0:

$$|\boldsymbol{F}| = \begin{cases} k(\ell_0 - \ell) & \text{if } \ell < \ell_0, \\ 0 & \text{if } \ell \geq \ell_0. \end{cases}$$

Get expanding mesh by choosing ℓ_0 larger than desired length h

Reactions at Boundaries



For each *boundary* node:

$$\sum_i \boldsymbol{F}_i + \boldsymbol{R} = 0$$

Reaction force $\boldsymbol{R}$:

- Orthogonal to boundary
- Keeps node along boundary

Node Movement and Connectivity Updates

- Move nodes $\boldsymbol{p}$ to find force equilibrium:

$$\boldsymbol{p}_{n+1} = \boldsymbol{p}_n + \Delta t \boldsymbol{F}(\boldsymbol{p}_n)$$

- Project boundary nodes to $\phi(\boldsymbol{p}) = 0$
- Elements deform, change connectivity based on element quality or in-circle condition (Delaunay)