Chapter 5 – Initial-Value Problems for Ordinary Differential Equations

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Math 128A Numerical Analysis

Lipschitz Condition and Convexity

Definition

A function $f(t, y)$ is said to satisfy a *Lipschitz condition* in the variable y on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists with

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|,$$

whenever $(t, y_1), (t, y_2) \in D$. The constant L is called a *Lipschitz constant* for f.

Definition

A set $D \subset \mathbb{R}^2$ is said to be *convex* if whenever (t_1, y_1) and (t_2, y_2) belong to D and λ is in $[0, 1]$, the point $((1 - \lambda)t_1 + \lambda t_2, (1 - \lambda)y_1 + \lambda y_2)$ also belongs to D.

Existence and Uniqueness

Theorem

Suppose $f(t, y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L > 0$ exists with

$$\left|\frac{\partial f}{\partial y}(t, y)\right| \leq L, \quad \text{for all } (t, y) \in D,$$

then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L.

Theorem

Suppose that $D = \{(t, y) \mid a \leq t \leq b, -\infty < y < \infty\}$ and that $f(t, y)$ is continuous on D. If f satisfies a Lipschitz condition on D in the variable y, then the initial-value problem

$$y'(t) = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

has a unique solution $y(t)$ for $a \leq t \leq b$.

Well-Posedness

Definition

The initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

is said to be a *well-posed problem* if:

- A unique solution, $y(t)$, to the problem exists, and
- There exist constants $\varepsilon_0 > 0$ and $k > 0$ such that for any ε, with $\varepsilon_0 > \varepsilon > 0$, whenever $\delta(t)$ is continuous with $|\delta(t)| < \varepsilon$ for all t in $[a, b]$, and when $|\delta_0| < \varepsilon$, the initial-value problem

  $$\frac{dz}{dt} = f(t, z) + \delta(t), \quad a \le t \le b, \quad z(a) = \alpha + \delta_0,$$

  has a unique solution $z(t)$ that satisfies

  $$|z(t) - y(t)| < k\varepsilon \quad \text{for all } t \text{ in } [a, b].$$

Well-Posedness

Theorem

Suppose $D = \{(t, y) \mid a \leq t \leq b$ and $-\infty < y < \infty\}$. If f is continuous and satisfies a Lipschitz condition in the variable y on the set D, then the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

is well-posed.

Euler's Method

Suppose a well-posed initial-value problem is given:

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

Distribute mesh points equally throughout $[a, b]$:

$$t_i = a + ih, \quad \text{for each } i = 0, 1, 2, \ldots, N.$$

The *step size* $h = (b - a)/N = t_{i+1} - t_i$.

Euler's Method

Use Taylor's Theorem for $y(t)$:

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2} y''(\xi_i)$$

for $\xi_i \in (t_i, t_{i+1})$. Since $h = t_{i+1} - t_i$ and $y'(t_i) = f(t_i, y(t_i))$,

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i).$$

Neglecting the remainder term gives Euler's method for $w_i \approx y(t_i)$:

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + hf(t_i, w_i), \qquad i = 0, 1, \dots, N-1 \end{aligned}$$

The well-posedness implies that

$$f(t_i, w_i) \approx y'(t_i) = f(t_i, y(t_i))$$

Error Bound

Theorem

Suppose f is continuous and satisfies a Lipschitz condition with constant L on

$$D = \{(t, y) \mid a \leq t \leq b, -\infty < y < \infty\}$$

and that a constant M exists with

$$|y''(t)| \leq M, \quad \text{for all } t \in [a, b].$$

Let $y(t)$ denote the unique solution to the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

and $w_0, w_1, \ldots, w_n$ as in Euler's method. Then

$$|y(t_i) - w_i| \leq \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right].$$

Local Truncation Error

Definition

The difference method

$$w_0 = \alpha$$
$$w_{i+1} = w_i + h\phi(t_i, w_i)$$

has *local truncation error*

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\phi(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i),$$

for each $i = 0, 1, \dots, N-1$.

Higher-Order Taylor Methods

Consider initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha.$$

Expand $y(t)$ in nth Taylor polynomial about t_i, evaluated at t_{i+1}:

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(t_i) + \cdots \\ &\quad + \frac{h^n}{n!} y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i) \\ &= y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(t_i, y(t_i)) + \cdots \\ &\quad + \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)) \end{aligned}$$

for some $\xi_i \in (t_i, t_{i+1})$. Delete remainder term to obtain the Taylor method of order n.

Higher-Order Taylor Methods

Taylor Method of Order n

$$w_0 = \alpha$$
$$w_{i+1} = w_i + hT^{(n)}(t_i, w_i), \qquad i = 0, 1, \ldots, N-1$$

where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{(n-1)}}{n!} f^{(n-1)}(t_i, w_i)$$

Higher-Order Taylor Methods

Theorem

If Taylor's method of order n is used to approximate the solution to

$$y'(t) = f(t, y(t)), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

with step size h and if $y \in C^{n+1}[a, b]$, then the local truncation error is $O(h^n)$.

Taylor's Theorem in Two Variables

Theorem

Suppose $f(t, y)$ and partial derivatives up to order $n+1$ continuous on $D = \{(t, y) \mid a \le t \le b, c \le y \le d\}$, let $(t_0, y_0) \in D$. For $(t, y) \in D$, there is $\xi \in [t, t_0]$ and $\mu \in [y, y_0]$ with

$$\begin{aligned}
f(t,y) &= P_n(t,y) + R_n(t,y) \\
P_n(t,y) &= f(t_0,y_0) + \left[(t-t_0)\frac{\partial f}{\partial t}(t_0,y_0) + (y-y_0)\frac{\partial f}{\partial y}(t_0,y_0)\right] \\
&\quad + \left[\frac{(t-t_0)^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0,y_0) + (t-t_0)(y-y_0)\frac{\partial^2 f}{\partial t \partial y}(t_0,y_0)\right. \\
&\quad + \left.\frac{(y-y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0,y_0)\right] + \cdots \\
&\quad + \left[\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(t-t_0)^{n-j}(y-y_0)^j\frac{\partial^n f}{\partial t^{n-j}\partial y^j}(t_0,y_0)\right]
\end{aligned}$$

Taylor's Theorem in Two Variables

Theorem

(cont'd)

$$R_n(t, y) = \frac{1}{(n+1)!} \sum_{j=0}^{n+1} \binom{n+1}{j} (t - t_0)^{n+1-j} (y - y_0)^j \cdot \frac{\partial^{n+1} f}{\partial t^{n+1-j} \partial y^j}(\xi, \mu)$$

$P_n(t, y)$ is the nth *Taylor polynomial in two variables*.

Runge-Kutta Methods

- Obtain high-order accuracy of Taylor methods without knowledges of derivatives of f
- Determine a_1, α_1, β_1 such that

$$a_1 f(t + \alpha_1, y + \beta_1) \approx f(t, y) + \frac{h}{2} f'(t, y) = T^{(2)}(t, y).$$

  with $O(h^2)$ error.
- Since

$$f'(t, y) = \frac{df}{dt}(t, y) = \frac{\partial f}{\partial t}(t, y) + \frac{\partial f}{\partial y}(t, y) \cdot y'(t)$$

  and $y'(t) = f(t, y)$, we have

$$T^{(2)}(t, y) = f(t, y) + \frac{h}{2}\frac{\partial f}{\partial t}(t, y) + \frac{h}{2}\frac{\partial f}{\partial y}(t, y) \cdot f(t, y)$$

Runge-Kutta Methods

- Expand $f(t+\alpha_1, y+\beta_1)$ in 1st degree Taylor polynomial:

$$a_1 f(t+\alpha_1, y+\beta_1) = a_1 f(t,y) + a_1\alpha_1 \frac{\partial f}{\partial t}(t,y) + a_1\beta_1 \frac{\partial f}{\partial y}(t,y) + a_1 \cdot R_1(t+\alpha_1, y+\beta_1)$$

- Matching coefficients gives

$$a_1 = 1 \quad a_1\alpha_1 = \frac{h}{2}, \quad a_1\beta_1 = \frac{h}{2} f(t,y)$$

with unique solution

$$a_1 = 1, \quad \alpha_1 = \frac{h}{2}, \quad \beta_1 = \frac{h}{2} f(t,y)$$

Runge-Kutta Methods

- This gives

$$T^{(2)}(t,y) = f\left(t + \frac{h}{2}, y + \frac{h}{2} f(t,y)\right) - R_1\left(t + \frac{h}{2}, y + \frac{h}{2} f(t,y)\right)$$

with $R_1(\cdot,\cdot) = O(h^2)$

Midpoint Method

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + hf\left(t + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right), \qquad i = 0, 1, \ldots, N-1$$

Local truncation error of order two.

Runge-Kutta Methods

Runge-Kutta Order Four

$$
\begin{aligned}
w_0 &= \alpha \\
k_1 &= hf(t_i, w_i) \\
k_2 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right) \\
k_3 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right) \\
k_4 &= hf(t_{i+1}, w_i + k_3) \\
w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}
$$

Local truncation error $O(h^4)$

Runge-Kutta Order Four

MATLAB Implementation

```
function [t, w] = rk4(f, a, b, alpha, N)
% Solve ODE y'(t) = f(t, y(t)) using Runge-Kutta 4.

h = (b-a) / N;
t = (a:h:b)';
w = zeros(N+1, length(alpha));
w(1,:) = alpha(:)';
for i = 1:N
    k1 = h*f(t(i),       w(i,:));
    k2 = h*f(t(i) + h/2, w(i,:) + k1/2);
    k3 = h*f(t(i) + h/2, w(i,:) + k2/2);
    k4 = h*f(t(i) + h,   w(i,:) + k3);
    w(i+1,:) = w(i,:) + (k1 + 2*k2 + 2*k3 + k4)/6;
end
```

Multistep Methods

Definition

An *m-step multistep method* for solving the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

has a difference equation for approximate w_{i+1} at t_{i+1}:

$$\begin{aligned} w_{i+1} = {} & a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} \\ & + h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots \\ & + b_0 f(t_{i+1-m}, w_{i+1-m})], \end{aligned}$$

where $h = (b-a)/N$, and starting values are specified:

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad \ldots, \quad w_{m-1} = \alpha_{m-1}$$

Explicit method if $b_m = 0$, *implicit* method if $b_m \neq 0$.

Multistep Methods

Fourth-Order Adams-Bashforth Technique

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3,$$

$$w_{i+1} = w_i + \frac{h}{24}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})]$$

Fourth-Order Adams-Moulton Technique

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2,$$

$$w_{i+1} = w_i + \frac{h}{24}[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})]$$

Derivation of Multistep Methods

Integrate the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

over $[t_i, t_{i+1}]$:

$$y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t))\, dt$$

Replace f by polynomial $P(t)$ interpolating $(t_0, w_0), \ldots, (t_i, w_i)$, and approximate $y(t_i) \approx w_i$:

$$y(t_{i+1}) \approx w_i + \int_{t_i}^{t_{i+1}} P(t)\, dt$$

Derivation of Multistep Methods

Adams-Bashforth explicit m-step: Newton backward-difference polynomial through
$(t_i, f(t_i, y(t_i))), \ldots, (t_{i+1-m}, f(t_{i+1-m}, y(t_{i+1-m})))$.

$$\begin{aligned}\int_{t_i}^{t_{i+1}} f(t, y(t))\,dt &\approx \int_{t_i}^{t_{i+1}} \sum_{k=0}^{m-1} (-1)^k \binom{-s}{k} \nabla^k f(t_i, y(t_i))\,dt \\ &= \sum_{k=0}^{m-1} \nabla^k f(t_i, y(t_i)) h (-1)^k \int_0^1 \binom{-s}{k}\,ds\end{aligned}$$

k	0	1	2	3	4	5
$(-1)^k \int_0^1 \binom{-s}{k}\,ds$	1	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{3}{8}$	$\frac{251}{720}$	$\frac{95}{288}$

Derivation of Multistep Methods

Three-step Adams-Bashforth:

$$y(t_{i+1}) \approx y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}\nabla f(t_i, y(t_i)) + \frac{5}{12}\nabla^2 f(t_i, y(t_i))\right]$$
$$= y(t_i) + \frac{h}{12}[23f(t_i, y(t_i)) - 16f(t_{i-1}, y(t_{i-1})) + 5f(t_{i-2}, y(t_{i-2}))]$$

The method is:

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2,$$
$$w_{i+1} = w_i + \frac{h}{12}[23f(t_i, w_i) - 16f(t_{i-1}, w_{i-1}) + 5f(t_{i-2}, w_{i-2})]$$

Local Truncation Error

Definition

If $y(t)$ solves

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

and

$$\begin{aligned} w_{i+1} = & a_{m-1} w_i + \cdots + a_0 w_{i+1-m} \\ & + h[b_m f(t_{i+1}, w_{i+1}) + \cdots + b_0 f(t_{i+1-m}, w_{i+1-m})], \end{aligned}$$

the *local truncation error* is

$$\begin{aligned} \tau_{i+1}(h) = & \frac{y(t_{i+1}) - a_{m-1} y(t_i) - \cdots - a_0 y(t_{i+1-m})}{h} \\ & - [b_m f(t_{i+1}, y(t_{i+1})) + \cdots + b_0 f(t_{i+1-m}, y(t_{i+1-m}))]. \end{aligned}$$

High-Order Systems of Initial-Value Problems

An m*th-order system* of first-order initial-value problems has the form

$$\begin{aligned}
\frac{du_1}{dt}(t) &= f_1(t, u_1, u_2, \ldots, u_m), \\
\frac{du_2}{dt}(t) &= f_2(t, u_1, u_2, \ldots, u_m), \\
&\vdots \\
\frac{du_m}{dt}(t) &= f_m(t, u_1, u_2, \ldots, u_m),
\end{aligned}$$

for $a \leq t \leq b$, with the initial conditions

$$u_1(a) = \alpha_1, u_2(a) = \alpha_2, \ldots, u_m(a) = \alpha_m.$$

Existence and Uniqueness

Definition

The function $f(t, y_1, \ldots, y_m)$, defined on the set

$$D = \{(t, u_1, \ldots, u_m) \mid a \leq t \leq b, -\infty < u_i < \infty, i = 1, 2, \ldots, m\}$$

is said to satisfy a *Lipschitz condition* on D in the variables $u_1, u_2, \ldots, u_m$ if a constant $L > 0$ exists with

$$|f(t, u_1, \ldots, u_m) - f(t, z_1, \ldots, z_m)| \leq L \sum_{j=1}^{m} |u_j - z_j|,$$

for all $(t, u_1, \ldots, u_m)$ and $(t, z_1, \ldots, z_m)$ in D.

Existence and Uniqueness

Theorem

Suppose

$$D = \{(t, u_1, \dots, u_m) \mid a \le t \le b, -\infty < u_i < \infty, i = 1, 2, \dots, m\}$$

and let $f_i(t, u_1, \dots, u_m)$, for each $i = 1, 2, \dots, m$, be continuous on D and satisfy a Lipschitz condition there. The system of first-order differential equations

$$\frac{du_k}{dt}(t) = f_k(t, u_1, \dots, u_m), \quad u_k(a) = \alpha_k, \quad k = 1, \dots, m$$

has a unique solution $u_1(t), \dots, u_m(t)$ for $a \le t \le b$.

Numerical Methods

Numerical methods for systems of first-order differential equations are vector-valued generalizations of methods for single equations.

Fourth order Runge-Kutta for systems

$$
\begin{aligned}
\mathbf{w}_0 &= \\
\mathbf{k}_1 &= h\mathbf{f}(t_i, \mathbf{w}_i) \\
\mathbf{k}_2 &= h\mathbf{f}(t_i + \frac{h}{2}, \mathbf{w}_i + \frac{1}{2}\mathbf{k}_1) \\
\mathbf{k}_3 &= h\mathbf{f}(t_i + \frac{h}{2}, \mathbf{w}_i + \frac{1}{2}\mathbf{k}_2) \\
\mathbf{k}_4 &= h\mathbf{f}(t_{i+1}, \mathbf{w}_i + \mathbf{k}_3) \\
\mathbf{w}_{i+1} &= \mathbf{w}_i + \frac{1}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4)
\end{aligned}
$$

where $\mathbf{w}_i = (w_{i,1}, \dots, w_{i,m})$ is the vector of unknowns.

Consistency and Convergence

Definition

A one-step difference-equation with local truncation error $\tau_i(h)$ is said to be *consistent* if

$$\lim_{h \to 0} \max_{1 \le i \le N} |\tau_i(h)| = 0$$

Definition

A one-step difference equation is said to be *convergent* if

$$\lim_{h \to 0} \max_{1 \le i \le N} |w_i - y(t_i)| = 0,$$

where $y_i = y(t_i)$ is the exact solution and w_i the approximation.

Convergence of One-Step Methods

Theorem

Suppose the initial-value problem $y' = f(t, y)$, $a \le t \le b$, $y(a) = \alpha$ is approximated by a one-step difference method in the form $w_0 = \alpha$, $w_{i+1} = w_i + h\phi(t_i, w_i, h)$. Suppose also that $h_0 > 0$ exists and $\phi(t, w, h)$ is continuous with a Lipschitz condition in w with constant L on D, then:

$$D = \{(t, w, h) \mid a \le t \le b, -\infty < w < \infty, 0 \le h \le h_0\}.$$

1. The method is stable;
2. The method is convergent if and only if it is consistent:

$$\phi(t, y, 0) = f(t, y)$$

3. If τ exists s.t. $|\tau_i(h)| \le \tau(h)$ when $0 \le h \le h_0$, then

$$|y(t_i) - w_i| \le \frac{\tau(h)}{L} e^{L(t_i - a)}.$$

Root Condition

Definition

Let $\lambda_1, \ldots, \lambda_m$ denote the roots of the characteristic equation

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - \cdots - a_1\lambda - a_0 = 0$$

associated with the multistep method

$$\begin{aligned} w_{i+1} =& a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} \\ & + hF(t_i, h, w_{i+1}, w_i, \ldots, w_{i+1-m}). \end{aligned}$$

If $|\lambda_i| \leq 1$ and all roots with absolute value 1 are simple, the method is said to satisfy the *root condition*.

Stability

Definition

1. Methods that satisfy the root condition and have $\lambda = 1$ as the only root of magnitude one are called *strongly stable*.
2. Methods that satisfy the root condition and have more than one distinct root with magnitude one are called *weakly stable*.
3. Methods that do not satisfy the root condition are *unstable*.

Theorem

A multistep method

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} + hF(t_i, h, w_{i+1}, w_i, \ldots, w_{i+1-m})$$

is stable if and and only if it satisfies the root condition. If it is also consistent, then it is stable if and only if it is convergent.

Stiff Equations

- A *stiff differential equation* is numerically unstable unless the step size is extremely small
- Large derivatives give error terms that are dominating the solution
- Example: The initial-value problem

$$y' = -30y, \quad 0 \le t \le 1.5, \quad y(0) = \frac{1}{3}$$

has exact solution $y = \frac{1}{3}e^{-30t}$. But RK4 is unstable with step size $h = 0.1$.

Euler's Method for Test Equation

- Consider the simple test equation

$$y' = \lambda y, \quad y(0) = \alpha, \quad \text{where } \lambda < 0$$

  with solution $y(t) = \alpha e^{\lambda t}$.
- Euler's method gives $w_0 = \alpha$ and

$$w_{j+1} = w_j + h(\lambda w_j) = (1 + h\lambda) w_j = (1 + h\lambda)^{j+1} \alpha.$$

- The absolute error is

$$\begin{aligned} |y(t_j) - w_j| &= \left| e^{jh\lambda} - (1 + h\lambda)^j \right| |\alpha| \\ &= \left| (e^{h\lambda})^j - (1 + h\lambda)^j \right| |\alpha| \end{aligned}$$

- Stability requires $|1 + h\lambda| < 1$, or $h < 2/|\lambda|$.

Multistep Methods

Apply a multistep method to the test equation:

$$\begin{aligned} w_{j+1} = & a_{m-1} w_j + \cdots + a_0 w_{j+1-m} \\ & + h\lambda (b_m w_{j+1} + b_{m-1} w_j + \cdots + b_0 w_{j+1-m}) \end{aligned}$$

or

$$(1 - h\lambda b_m) w_{j+1} - (a_{m-1} + h\lambda b_{m-1}) w_j - \cdots - (a_0 + h\lambda b_0) w_{j+1-m} = 0$$

Let $\beta_1, \ldots, \beta_m$ be the zeros of the *characteristic polynomial*

$$Q(z, h\lambda) = (1 - h\lambda b_m) z^m - (a_{m-1} + h\lambda b_{m-1}) z^{m-1} - \cdots - (a_0 + h\lambda b_0)$$

Then $c_1, \ldots, c_m$ exist with

$$w_j = \sum_{k=1}^{m} c_k (\beta_k)^j$$

and $|\beta_k| < 1$ is required for stability.

Region of Stability

Definition

The *region R of absolute stability* for a one-step method is $R = \{h\lambda \in \mathcal{C} \mid |Q(h\lambda)| < 1\}$, and for a multistep method, it is $R = \{h\lambda \in \mathcal{C} \mid |\beta_k| < 1, \text{ for all zeros } \beta_k \text{ of } Q(z, h\lambda)\}$.

A numerical method is said to be *A-stable* if its region R of absolute stability contains the entire left half-plane.