Low-temperature random matrix theory at the soft edge

Alan Edelman

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Per-Olof Persson

Department of Mathematics, University of California, Berkeley, CA 94720, USA

Brian D. Sutton

Department of Mathematics, Randolph-Macon College, Ashland, VA 23005, USA

Abstract

"Low temperature" random matrix theory is the study of random eigenvalues as energy is removed. In standard notation, β is identified with inverse temperature, and low temperatures are achieved through the limit $\beta \to \infty$. In this paper, we derive statistics for low-temperature random matrices at the "soft edge," which describes the extreme eigenvalues for many random matrix distributions. Specifically, new asymptotics are found for the expected value and standard deviation of the general-β Tracy-Widom distribution. The new techniques utilize beta ensembles, stochastic differential operators, and Riccati diffusions. The asymptotics fit known high-temperature statistics curiously well.

I. INTRODUCTION

With modern technology, a random matrix can be "frozen," leaving a deterministic object that is often easier to study. As heat is reapplied, the matrix thaws, and the effects of randomness can be studied in a new way. This article focuses on the asymptotic regime $\beta \to \infty$, reviewing prior work on finite beta ensembles and presenting new results on stochastic differential operators and Riccati diffusions. In the end, we gain a better understanding of the low-temperature regime and introduce methods that may apply generally to all $\beta > 0$.

With the publication of Dyson's "Statistical Theory of the Energy Levels of Complex Systems" in 1962 [10], a bifurcation of random matrix theory became inevitable. Dyson showed that the eigenvalues of random matrices obey the laws of a "log gas" from statistical mechanics, with the division algebra of matrix entries determining the temperature of the gas. Specifically, the parameter β equaled inverse temperature: $\beta = \frac{1}{kT}$. While the random matrices seemed limited to three classes, as Dyson himself emphasized in his "Threefold Way" paper of the same year, the eigenvalues generalized naturally to any $\beta > 0$. A researcher might choose to study random matrices at $\beta = 1, 2, 4$ or their eigenvalues for any $\beta > 0$, but the choice would likely lead to totally different methods.

In more recent times, classical and general-β random matrix theory have been reunited by tridiagonal beta ensembles, stochastic differential operators, and Riccati diffusions. Tridiagonal beta ensembles, introduced by Dumitriu and Edelman [8], extend random matrices to general $\beta > 0$. Stochastic differential operators, introduced by Alan Edelman at the 2003 *SIAM Conference on Applied Linear Algebra* [11] and developed by Edelman and Sutton [12] and Ramírez, Rider, and Virág [15], are the $n \to \infty$ continuum limits of random matrices. Riccati diffusions for beta ensembles, introduced by Ramírez, Rider, and Virág [15], and the equivalent Sturm sequence characterization, discovered independently by Albrecht, Chan, and Edelman [1], transform the eigenvalue problems from second-order differential equations to first-order diffusion processes. With these new tools, we can work directly with general-β operators rather than disembodied general-β eigenvalues. Still, there is much work to be done. Although the operators have been extended, few classical methods have made the transition. In this article, we develop new methods for working in the $\beta \to \infty$ regime.

Specifically, we investigate two asymptotic expressions for the *soft edge* as $\beta \to \infty$. The soft edge can be investigated by taking an $n \to \infty$ limit of many different random matrix

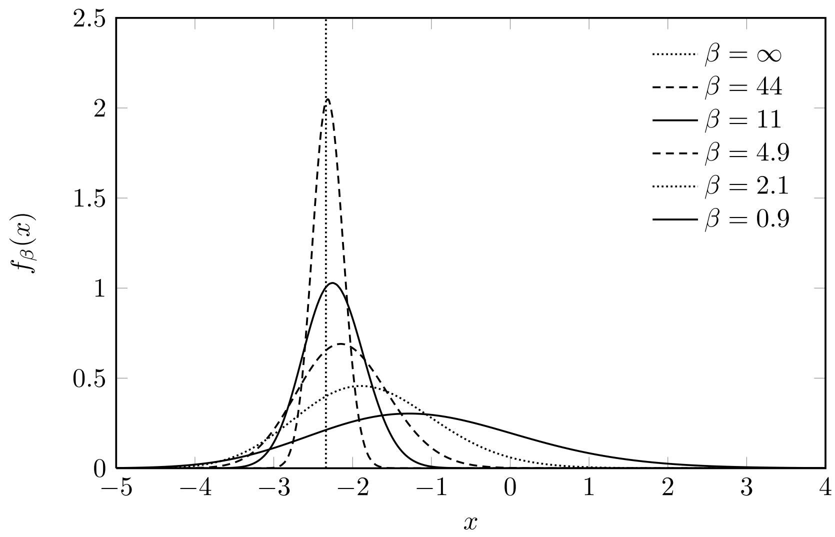

Figure 1. Largest eigenvalue for several values of β

distributions. For a concrete case, let G be an n-by-n matrix with iid real standard Gaussian entries, and let $H = \frac{1}{2}(G+G^T)$. Then H is a random symmetric matrix, and its distribution is called the *Gaussian orthogonal ensemble* (GOE) or the *Hermite ensemble* with $\beta = 1$. The soft edge appears as n approaches ∞ and the spectrum is recentered and rescaled to illuminate the largest eigenvalue. The limiting eigenvalue distribution is often called the *Tracy-Widom distribution* with $\beta = 1$ [17]. If G has complex or quaternion entries, then the Tracy-Widom distribution with $\beta = 2$ or $\beta = 4$ is realized [18].

Beta ensembles generalize the $\beta = 1, 2, 4$ triad to arbitrary $\beta > 0$. Tracy-Widom distributions for several values of β are shown in Figure 1. They were computed with the numerical routine of Bloemendal and Sutton [2].

We argue the following two asymptotic statistics, in which λ_k^β is the centered and scaled kth eigenvalue at the soft edge, Ai is the bounded Airy function, a_k is the kth zero of Ai, and $G(x,x)$ is the diagonal of a Green's function defined in (7):

$$\mathrm{SD}[\lambda_k^\beta] \sim \frac{2}{\sqrt{\beta}} \sqrt{\int_0^\infty \left(\frac{1}{\mathrm{Ai}'(a_k)} \mathrm{Ai}(x + a_k) \right)^4 dx}, \quad \beta \to \infty, \tag{1}$$

$$\mathrm{E}[\lambda_k^\beta] \sim a_k - \frac{4}{\beta} \int_0^\infty G(x,x) \left(\frac{1}{\mathrm{Ai}'(a_k)} \mathrm{Ai}(x + a_k) \right)^2 dx, \quad \beta \to \infty. \tag{2}$$

Our methods support error terms of $O(\beta^{-1})$ and $O(\beta^{-3/2})$ for (1) and (2), respectively.

For $k = 1$, we have the following numerics:

$$\mathrm{SD}[\lambda_1^\beta] \sim 0.64609\,09360 \times \frac{2}{\sqrt{\beta}}, \quad \beta \to \infty, \tag{3}$$

$$\mathrm{E}[\lambda_1^\beta] \sim -2.33810\,74105 + 0.28120\,34761 \left(\frac{2}{\sqrt{\beta}}\right)^2, \quad \beta \to \infty. \tag{4}$$

These are plotted in Figure 2 and compared with known statistics for $\beta = 1, 2, 4, \infty$ [5]:

β	Mean	Asymptotic prediction
∞	−2.33811	−2.33811
4	−2.05520	−2.05690
2	−1.77109	−1.77570
1	−1.20653	−1.21329

β	Standard deviation	Asymptotic prediction
∞	0	0
4	0.64103	0.64609
2	0.90177	0.91371
1	1.26798	1.29218

Our normalization at $\beta = 4$ differs from that of Tracy and Widom by a factor of $2^{1/6}$ [12]. The asymptotics fit the known values remarkably well.

The ideas behind (1) and (2) extend naturally to higher moments and higher-order asymptotics. We feel that their potential for further development is as exciting as the surprisingly good fits of Figure 2.

Equation (1) was found earlier by Dumitriu and Edelman [9]. Their argument, based on finite-dimensional general-β matrix models, is reviewed in Section II. Two new arguments, based on stochastic differential operators and Riccati diffusions, are given in Sections III and IV, respectively. Equation (2) is derived for the first time in Section III. The stochastic operator and Riccati diffusion methods appear to be novel.

II. FINITE BETA ENSEMBLES

Dumitriu and Edelman previously derived expression (1) for the standard deviation using their *beta ensembles* [9]. This section reviews their argument.

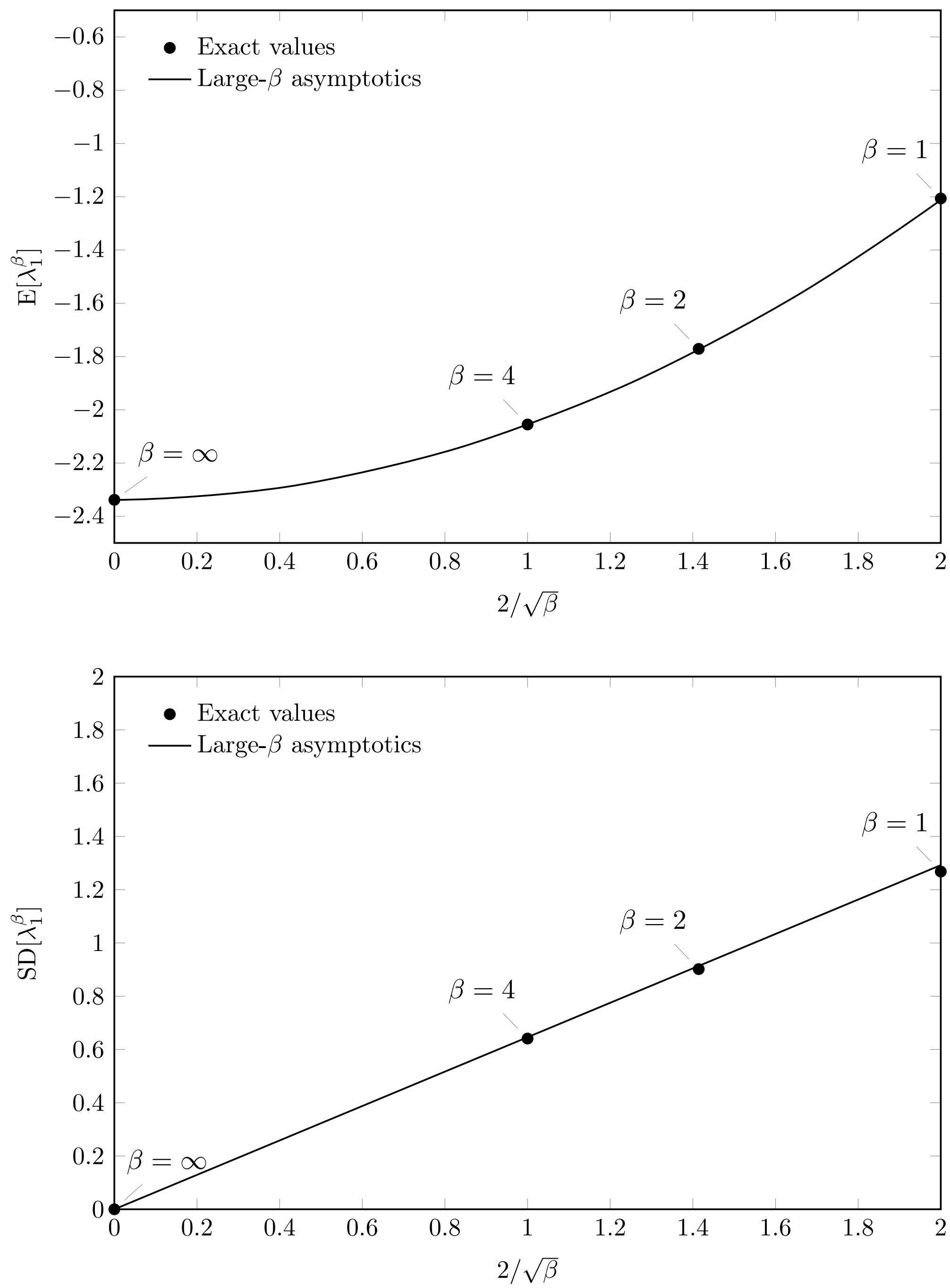

Figure 2. Mean and standard deviation asymptotics at the soft edge

The Gaussian orthogonal, unitary, and symplectic ensembles have joint eigenvalue density

$$\text{const} \times e^{-(\beta/2)\sum_{i=1}^{n}\lambda_i^2} \prod_{1\le i<j\le n} |\lambda_i - \lambda_j|^\beta \tag{5}$$

for $\beta = 1, 2, 4$, respectively. The β-*Hermite ensemble* is the n-by-n random symmetric tridiagonal matrix

$$H^\beta \sim \frac{1}{\sqrt{2\beta}} \begin{bmatrix} \sqrt{2}G_1 & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & \sqrt{2}G_2 & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & \sqrt{2}G_{n-1} & \chi_\beta \\ & & & \chi_\beta & \sqrt{2}G_n \end{bmatrix}, \tag{6}$$

in which $G_1, \dots, G_n$ are standard Gaussian variables, χ_r is a chi-distributed random variable with r degrees of freedom, and all entries in the upper-triangular part are independent [8]. The random tridiagonal has the eigenvalue density (5) for *all* positive β. The ensemble is extended further by defining

$$H^\infty = \lim_{\beta\to\infty} H^\beta = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & \sqrt{n-1} & & & \\ \sqrt{n-1} & 0 & \sqrt{n-2} & & \\ & \ddots & \ddots & \ddots & \\ & & \sqrt{2} & 0 & \sqrt{1} \\ & & & \sqrt{1} & 0 \end{bmatrix}.$$

This deterministic matrix encodes the three-term recurrence for Hermite polynomials. Its eigenvalues are the roots $h_1 > h_2 > \cdots > h_n$ of the nth Hermite polynomial H_n, and its kth eigenvector is

$$v_k = (H_{n-1}(h_k), H_{n-2}(h_k), \dots, H_1(h_k), H_0(h_k)).$$

Large-β asymptotics at the soft edge concern the largest eigenvalues of H^β as $\beta \to \infty$ and $n \to \infty$. Dumitriu and Edelman let $\beta \to \infty$ first and $n \to \infty$ second.

First, they show

$$\lim_{\beta\to\infty} \sqrt{\beta}(H^\beta - H^\infty) = Z$$

almost surely, in which Z is a symmetric tridiagonal matrix with independent mean-zero Gaussian entries having standard deviation 1 on the diagonal and 1/2 on the superdiagonal. Essentially,

$$H^\beta \sim H^\infty + \frac{1}{\sqrt{\beta}} Z, \quad \beta \to \infty.$$

When β is large, eigenvalue perturbation theory applies. The kth eigenvalue $\lambda_k(H^\beta)$ satisfies

$$\lambda_k(H^\beta) \sim h_k + \frac{1}{\sqrt{\beta}} \frac{v_k^T Z v_k}{v_k^T v_k}, \quad \beta \to \infty.$$

Next, n approaches ∞. In this limit, the largest eigenvalue tends toward infinity, while its nearest neighbor becomes arbitrarily close. To see the eigenvalue more clearly, a recentering and rescaling are necessary:

$$\sqrt{2} n^{1/6} (\lambda_k(H^\beta) - \sqrt{2n}) \sim \sqrt{2} n^{1/6} (h_k - \sqrt{2n}) + \sqrt{2} n^{1/6} \frac{1}{\sqrt{\beta}} \frac{v_k^T Z v_k}{v_k^T v_k}, \quad \beta \to \infty.$$

The left-hand side converges in distribution to the general-β Tracy-Widom distribution. Using orthogonal polynomial asymptotics, Dumitriu and Edelman show that the right-hand side converges to a Gaussian whose mean is the Airy zero a_k and whose standard deviation is given by (1).

III. STOCHASTIC DIFFERENTIAL OPERATORS

In this section, we compute the mean eigenvalue asymptotics (2) for the first time and rederive the standard deviation result (1). Our method is based on the stochastic operator approach.

The stochastic operator approach works with $n \to \infty$ continuum limits of random matrices, rather than the limiting eigenvalue distributions alone. In particular, the Hermite ensemble H^β of (6) has a continuum limit when scaled at the soft edge [12]:

$$\sqrt{2} n^{1/6} (H_\beta - \sqrt{2n}\, I) \xrightarrow{n \to \infty} \mathcal{A}^\beta,$$

in which $\mathcal{A}^\beta$ is the *stochastic Airy operator*

$$\boxed{\mathcal{A}^\beta = \frac{d^2}{dx^2} - x + \frac{2}{\sqrt{\beta}} W_x, \qquad \text{b.c.'s } f(0) = f(+\infty) = 0.}$$

W_x denotes a diagonal white noise process, so that $\int_a^b W_x f(x) dx = \int_a^b f(x) dB_x$ is a stochastic integral. The continuum limit is justified by viewing the tridiagonal matrix as a finite difference approximation of the continuous operator [12, 15]. In particular, the first k eigenvectors of H^β sample the first k eigenfunctions of $\mathcal{A}^\beta$ on a grid.

When $\beta = \infty$, the stochastic Airy operator becomes just the Airy operator $\mathcal{A}^\infty = \frac{d^2}{dx^2} - x$. Its eigenvalues are the zeros of the Airy function a_k, and its eigenvectors are $\frac{1}{\mathrm{Ai}'(a_k)}\,\mathrm{Ai}(x+a_k)$, $k = 1, 2, 3, \ldots$.

Our plan is to defrost the deterministic Airy operator and study $\mathcal{A}^\beta = \frac{d^2}{dx^2} - x + \varepsilon W_x$ for a small $\varepsilon = \frac{2}{\sqrt{\beta}}$. This approach appeared for the first time in Sutton's Ph.D. thesis [16].

A. Eigenvalue perturbation theory

Express the kth eigenvalue λ_k^β and its associated eigenfunction v_k^β in asymptotic series:

$$\lambda_k^\beta = \lambda^{(0)} + \varepsilon\lambda^{(1)} + \varepsilon^2\lambda^{(2)} + \varepsilon^3\lambda^{(3)} + \cdots,$$
$$v_k^\beta = v^{(0)} + \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \varepsilon^3 v^{(3)} + \cdots.$$

Eigenvalue perturbation theory yields

$$\begin{aligned}
\lambda^{(0)} &= a_k, & v^{(0)} &= \frac{1}{\mathrm{Ai}'(a_k)}\,\mathrm{Ai}(x + a_k),\\
\lambda^{(1)} &= \langle v^{(0)}, W_x v^{(0)}\rangle, & v^{(1)} &= (\mathcal{A}_\infty - a_k)^+\,(-W_x v^{(0)}),\\
\lambda^{(2)} &= \langle v^{(0)}, W_x v^{(1)}\rangle, & v^{(2)} &= (\mathcal{A}_\infty - a_k)^+\,(-W_x v^{(1)} + \lambda^{(1)} v^{(1)}),\\
\lambda^{(3)} &= \langle v^{(0)}, W_x v^{(2)}\rangle, & &\ldots,
\end{aligned}$$

with $(\mathcal{L}-\lambda)^+$ denoting the Moore-Penrose pseudoinverse of $\mathcal{L}-\lambda$, a.k.a. the reduced resolvent of $\mathcal{L}$ with respect to λ [13, II-§2.2].

B. Green's function

The specific pseudoinverse $(\mathcal{A}^\infty - a_k)^+$ is an integral operator whose kernel is a generalized Green's function. In the following formula, Bi is the second standard solution of Airy's equation [7, 9.8.i]:

$$
\boxed{
\begin{aligned}
[(\mathcal{A}^\infty - a_k)^+ f](x) &= \int_0^\infty G(x,y) f(y) dy, \\
G(x,y) &= -\frac{\operatorname{Ai}(x+a_k)\operatorname{Ai}'(y+a_k) + \operatorname{Ai}'(x+a_k)\operatorname{Ai}(y+a_k)}{\operatorname{Ai}'(a_k)^2} \\
&\qquad + \frac{\pi\operatorname{Bi}'(a_k)}{\operatorname{Ai}'(a_k)}\operatorname{Ai}(x+a_k)\operatorname{Ai}(y+a_k) \\
&\qquad - \pi \begin{cases} \operatorname{Bi}(x+a_k)\operatorname{Ai}(y+a_k), & x \le y \\ \operatorname{Ai}(x+a_k)\operatorname{Bi}(y+a_k), & x > y \end{cases}.
\end{aligned}
} \tag{7}
$$

Hence, $(\mathcal{A}^\infty - a_k)(\mathcal{A}^\infty - a_k)^+ f = (\mathcal{A}^\infty - a_k)^+(\mathcal{A}^\infty - a_k) f = f - \left\langle v^{(0)}, f \right\rangle v^{(0)}$ for every sufficiently smooth f.

To prove (7), we show the following [6, V–§14]:

1. G is symmetric,
2. G satisfies the boundary conditions $G(0,y) = \lim_{x\to+\infty} G(x,y) = 0$,
3. $(\mathcal{A}^\infty - a_k)G = -v_k(x)v_k(y)$ for $x \neq y$, in which $v_k(x)$ is the eigenvector $\frac{1}{\operatorname{Ai}'(a_k)}\operatorname{Ai}(x+a_k)$,
4. $\lim_{\delta\to 0} \frac{dG}{dx}\Big|_{x=y-\delta}^{y+\delta} = 1$, and
5. $\int_0^\infty G(x,y)\left(\frac{1}{\operatorname{Ai}'(a_k)}\operatorname{Ai}(x+a_k)\right) dx = 0$.

G is symmetric by inspection.

At $x = 0$, we find

$$
\begin{aligned}
&G(0,y) = -\tfrac{1}{\operatorname{Ai}'(a_k)^2}\operatorname{Ai}'(a_k)\operatorname{Ai}(y+a_k) - \pi\operatorname{Bi}(a_k)\operatorname{Ai}(y+a_k) \\
&= -\tfrac{1}{\operatorname{Ai}'(a_k)}\left[1 + \pi\operatorname{Ai}'(a_k)\operatorname{Bi}(a_k)\right]\operatorname{Ai}(y+a_k).
\end{aligned}
$$

From the Wronskian $\mathcal{W}\{\operatorname{Ai},\operatorname{Bi}\} = \operatorname{Ai}(x)\operatorname{Bi}'(x) - \operatorname{Ai}'(x)\operatorname{Bi}(x) = \frac{1}{\pi}$ [7, 9.2.E7], we find $\frac{1}{\pi} = \operatorname{Ai}(a_k)\operatorname{Bi}'(a_k) - \operatorname{Ai}'(a_k)\operatorname{Bi}(a_k) = -\operatorname{Ai}'(a_k)\operatorname{Bi}(a_k)$, which shows $G(0,y) = 0$. At the other boundary, $\lim_{x\to+\infty} G(x,y) = 0$ because Ai and Ai′ decay at infinity.

Away from the diagonal,

$$\frac{d^2G}{dx^2} = -\tfrac{1}{\mathrm{Ai}'(a_k)^2}(x+a_k)\,\mathrm{Ai}(x+a_k)\,\mathrm{Ai}'(y+a_k)$$
$$- \tfrac{1}{\mathrm{Ai}'(a_k)^2}(\mathrm{Ai}(x+a_k) + (x+a_k)\,\mathrm{Ai}'(x+a_k))\,\mathrm{Ai}(y+a_k)$$
$$+ \tfrac{\pi\,\mathrm{Bi}'(a_k)}{\mathrm{Ai}'(a_k)}(x+a_k)\,\mathrm{Ai}(x+a_k)\,\mathrm{Ai}(y+a_k)$$
$$- \pi \begin{cases} (x+a_k)\,\mathrm{Bi}(x+a_k)\,\mathrm{Ai}(y+a_k), & x<y \\ (x+a_k)\,\mathrm{Ai}(x+a_k)\,\mathrm{Bi}(y+a_k), & x>y \end{cases}$$

and

$$(x+a_k)G = -(x+a_k)\tfrac{1}{\mathrm{Ai}'(a_k)^2}\,\mathrm{Ai}(x+a_k)\,\mathrm{Ai}'(y+a_k)$$
$$- (x+a_k)\tfrac{1}{\mathrm{Ai}'(a_k)^2}\,\mathrm{Ai}'(x+a_k)\,\mathrm{Ai}(y+a_k)$$
$$+ \tfrac{\pi\,\mathrm{Bi}'(a_k)}{\mathrm{Ai}'(a_k)}(x+a_k)\,\mathrm{Ai}(x+a_k)\,\mathrm{Ai}(y+a_k)$$
$$- \pi \begin{cases} (x+a_k)\,\mathrm{Bi}(x+a_k)\,\mathrm{Ai}(y+a_k), & x\le y \\ (x+a_k)\,\mathrm{Ai}(x+a_k)\,\mathrm{Bi}(y+a_k), & x>y \end{cases},$$

and so

$$(\mathcal{A}^\infty - a_k)G = \frac{d^2G}{dx^2} - (x+a_k)G = -\tfrac{1}{\mathrm{Ai}'(a_k)^2}\,\mathrm{Ai}(x+a_k)\,\mathrm{Ai}(y+a_k), \quad x \ne y.$$

Looking for a jump discontinuity, we compute

$$\frac{dG}{dx} = -\tfrac{1}{\mathrm{Ai}'(a_k)^2}\,\mathrm{Ai}'(x+a_k)\,\mathrm{Ai}'(y+a_k)$$
$$- \tfrac{1}{\mathrm{Ai}'(a_k)^2}(x+a_k)\,\mathrm{Ai}(x+a_k)\,\mathrm{Ai}(y+a_k)$$
$$+ \tfrac{\pi\,\mathrm{Bi}'(a_k)}{\mathrm{Ai}'(a_k)}\,\mathrm{Ai}'(x+a_k)\,\mathrm{Ai}(y+a_k)$$
$$- \pi \begin{cases} \mathrm{Bi}'(x+a_k)\,\mathrm{Ai}(y+a_k), & x\le y \\ \mathrm{Ai}'(x+a_k)\,\mathrm{Bi}(y+a_k), & x>y \end{cases}$$

and find, using the Wronskian again,

$$\lim_{\delta\to 0} \frac{dG}{dx}\bigg|_{x=y-\delta}^{y+\delta} = -\pi\,\mathrm{Ai}'(y+a_k)\,\mathrm{Bi}(y+a_k) + \pi\,\mathrm{Bi}'(y+a_k)\,\mathrm{Ai}(y+a_k) = 0.$$

Finally, we want to show $\int_0^\infty G(x,y)\tfrac{1}{\mathrm{Ai}'(a_k)}\,\mathrm{Ai}(x+a_k)dx = 0$, or equivalently $\int_0^\infty G(x,y)\,\mathrm{Ai}(x+$

$a_k)dx = 0$. The following integral computations are straightforward [7, 9.11.iv]:

$$\int_0^\infty \operatorname{Ai}^2(x + a_k)dx = \operatorname{Ai}'(a_k)^2,$$
$$\int_y^\infty \operatorname{Ai}^2(x + a_k)dx = \left(\operatorname{Ai}'(y + a_k)^2 - (y + a_k)\operatorname{Ai}^2(y + a_k)\right),$$
$$\int_0^y \operatorname{Ai}(x + a_k)\operatorname{Bi}(x + a_k)dx = -\operatorname{Ai}'(y + a_k)\operatorname{Bi}'(y + a_k)$$
$$+ (y + a_k)\operatorname{Ai}(y + a_k)\operatorname{Bi}(y + a_k) + \operatorname{Ai}'(a_k)\operatorname{Bi}'(a_k),$$
$$\int_0^\infty \operatorname{Ai}(x + a_k)\operatorname{Ai}'(x + a_k)dx = 0.$$

The desired integral $\int_0^\infty G(x, y)\operatorname{Ai}(x + a_k)dx$ is the sum of the following five integrals:

$$\int_0^\infty -\tfrac{1}{\operatorname{Ai}'(a_k)^2}\operatorname{Ai}(x + a_k)\operatorname{Ai}'(y + a_k)\operatorname{Ai}(x + a_k)dx = -\operatorname{Ai}'(y + a_k),$$
$$\int_0^\infty -\tfrac{1}{\operatorname{Ai}'(a_k)^2}\operatorname{Ai}'(x + a_k)\operatorname{Ai}(y + a_k)\operatorname{Ai}(x + a_k)dx = 0,$$
$$\int_0^\infty \tfrac{\pi\operatorname{Bi}'(a_k)}{\operatorname{Ai}'(a_k)}\operatorname{Ai}(x+a_k)\operatorname{Ai}(y+a_k)\operatorname{Ai}(x+a_k)dx$$
$$= \pi\operatorname{Ai}'(a_k)\operatorname{Bi}'(a_k)\operatorname{Ai}(y+a_k),$$
$$\int_0^y -\pi\operatorname{Bi}(x + a_k)\operatorname{Ai}(y + a_k)\operatorname{Ai}(x + a_k)dx$$
$$= \pi\operatorname{Ai}(y + a_k)\operatorname{Ai}'(y + a_k)\operatorname{Bi}'(y + a_k)$$
$$- \pi(y + a_k)\operatorname{Ai}^2(y + a_k)\operatorname{Bi}(y + a_k)$$
$$- \pi\operatorname{Ai}'(a_k)\operatorname{Bi}'(a_k)\operatorname{Ai}(y + a_k),$$
$$\int_y^\infty -\pi\operatorname{Ai}(x + a_k)\operatorname{Bi}(y + a_k)\operatorname{Ai}(x + a_k)dx$$
$$= -\pi\operatorname{Ai}'(y + a_k)^2\operatorname{Bi}(y + a_k) + \pi(y + a_k)\operatorname{Ai}^2(y + a_k)\operatorname{Bi}(y + a_k)$$
$$= \operatorname{Ai}'(y + a_k) - \pi\operatorname{Ai}(y + a_k)\operatorname{Ai}'(y + a_k)\operatorname{Bi}'(y + a_k)$$
$$+ \pi(y + a_k)\operatorname{Ai}^2(y + a_k)\operatorname{Bi}(y + a_k).$$

All terms cancel, leaving $\int_0^\infty G(x, y)\operatorname{Ai}(x + a_k)dx = 0$. The Green's function is justified.

C. Eigenvalue asymptotics

Now the eigenvalue perturbation terms can be computed. At the order of $\varepsilon = \frac{2}{\sqrt{\beta}}$, the eigenvalue perturbation is Gaussian:

$$\lambda^{(1)} = \int_0^\infty \frac{1}{\mathrm{Ai}'(a_k)} \mathrm{Ai}(x+a_k) W_x \frac{1}{\mathrm{Ai}'(a_k)} \mathrm{Ai}(x+a_k) dx,$$

or more simply,

$$\boxed{\lambda^{(1)} = \int_0^\infty \left(\frac{1}{\mathrm{Ai}'(a_k)} \mathrm{Ai}(x+a_k) \right)^2 dB_x}$$

Its mean is 0 and its standard deviation is

$$\sigma = \sqrt{\int_0^\infty \left(\tfrac{1}{\mathrm{Ai}'(a_k)} \mathrm{Ai}(x+a_k) \right)^4 dx}.$$

The perturbation of the mean is on the order of $\varepsilon^2 = \left(\frac{2}{\sqrt{\beta}}\right)^2$. We find

$$\begin{aligned} v^{(1)}(x) &= \left[(\mathcal{A}^\infty - a_k)^+ \left(-W_x \tfrac{1}{\mathrm{Ai}'(a_k)} \mathrm{Ai}(x+a_k) \right) \right] \\ &= \int_0^\infty G(x,y) \left(-W_y \tfrac{1}{\mathrm{Ai}'(a_k)} \mathrm{Ai}(y+a_k) \right) dy \end{aligned}$$

and

$$\lambda^{(2)} = -\int_0^\infty \int_0^\infty \tfrac{1}{\mathrm{Ai}'(a_k)} \mathrm{Ai}(x+a_k) W_x G(x,y) W_y \tfrac{1}{\mathrm{Ai}'(a_k)} \mathrm{Ai}(y+a_k)\, dy\, dx,$$

or more simply,

$$\boxed{\lambda^{(2)} = -\int_0^\infty \int_0^\infty \frac{1}{\mathrm{Ai}'(a_k)^2} \mathrm{Ai}(x+a_k) G(x,y) \mathrm{Ai}(y+a_k) dB_y dB_x.}$$

The expected value is determined by the autocorrelation $\mathrm{E}[\iint f(x,y) dB_y dB_x] = \int f(x,x) dx$ (informally, $\mathrm{E}[W_x W_y] = \delta(x-y)$):

$$\mathrm{E}[\lambda^{(2)}] = -\int_0^\infty G(x,x) \left(\tfrac{1}{\mathrm{Ai}'(a_k)} \mathrm{Ai}(x+a_k) \right)^2 dx.$$

Asymptotic relations (1) and (2) follow.

IV. RICCATI DIFFUSION

The eigenvalue-eigenvector equation for the stochastic Airy operator is

$$\left(\frac{d^2}{dx^2} - x + \frac{2}{\sqrt{\beta}}W_x - \zeta\right) f = 0.$$

The Riccati transform $Y_x = \frac{f'(x)}{f(x)}$ produces the equivalent diffusion process

$$dY_x = (x + \zeta - Y_x^2)dx + \frac{2}{\sqrt{\beta}}dB_x,$$

and the boundary conditions $f(0) = f(+\infty) = 0$ translate to $Y_0 = +\infty$, $Y_x \sim \frac{\mathrm{Ai}'(x)}{\mathrm{Ai}(x)} \sim -\sqrt{x}$, $x \to +\infty$ [15].

In the eigenvalue-eigenvector equation, if ζ is to the right of all eigenvalues, then a solution $f(x)$ satisfying the left boundary condition will fail to meet the right boundary condition. Experiencing no sign changes and failing to decay, it will grow without bound on the same order as $\mathrm{Bi}(x)$ as $x \to +\infty$. The analogous statement for the Riccati diffusion is this: if ζ dominates all eigenvalues, then a solution Y_x to the diffusion process, started at $Y_0 = +\infty$, will have no poles in the $x > 0$ half-plane and will become asymptotic to $\frac{\mathrm{Bi}'(x)}{\mathrm{Bi}(x)} \sim \sqrt{x}$ as $x \to +\infty$. Vice versa, if ζ is to the left of some eigenvalue, then $f(x)$ satisfying $f(0) = 0$ will experience a sign change for some positive x, and Y_x satisfying $Y_0 = +\infty$ will reach $-\infty$ at the same finite x.

Thus, $\Pr[\lambda_1 < \zeta] = \Pr_{(0,+\infty)}[Y_x \text{ does not hit } -\infty]$, in which the notation indicates a diffusion started at $Y_0 = +\infty$ and run forward in time x. Equivalently, setting $t = x + \zeta$, we have

$$\boxed{dY_t = (t - Y_t^2)dt + \frac{2}{\sqrt{\beta}}dB_t}$$

and $\Pr[\lambda_1 < \zeta] = \Pr_{(\zeta,+\infty)}[Y_t \text{ does not hit } -\infty]$ as t runs from ζ to $+\infty$. The CDF $F(t) = F(t, \infty)$ of the rightmost eigenvalue will fall out as a special case after computing $F(t_0, y_0) = \Pr_{(t_0,y_0)}[Y_t \text{ does not hit } -\infty]$ for a general initial condition $Y_{t_0} = y_0$. Below, we write $F(t, y)$ instead of $F(t_0, y_0)$ to keep the notation clean.

The hitting probability has been investigated by Bloemendal and Virág [3, 4]. It is given by Kolmogorov's backward equation:

$$\frac{dF}{dt} + (t - y^2)\frac{\partial F}{\partial y} + \frac{2}{\beta}\frac{\partial^2 F}{\partial y^2} = 0,$$
$$F(+\infty, y) = 1,$$
$$F(t, y) \sim \Phi\left(\frac{\sqrt{\beta}}{2}\frac{t - y^2}{\sqrt{-y}}\right),\ y \to -\infty.$$

Φ denotes the CDF of the standard Gaussian distribution.

A. Amelioration as $\beta \to \infty$

This PDE has a simple solution when $\beta = \infty$, specifically $F(t, y) = \mathbf{1}[y > \frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}]$. In addition, it is well behaved for $\beta \ll \infty$. However, as $\beta \to \infty$, the equation is dominated by convection and the solution develops a region of rapid change around $y = \frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}$. The developing cliff breaks into a jump discontinuity once β reaches infinity.

To inspect the region of rapid change (the interesting region when $\beta \to \infty$) we apply a change of variables. Let $F(t, y) = \Phi(\frac{\sqrt{\beta}}{2}u(t, y))$ with $\Phi(z)$ the CDF of the standard Gaussian distribution. We have

$$\frac{dF}{dt} = \Phi'\left(\frac{\sqrt{\beta}}{2}u\right)\frac{\sqrt{\beta}}{2}\frac{du}{dt},$$
$$\frac{\partial F}{\partial y} = \Phi'\left(\frac{\sqrt{\beta}}{2}u\right)\frac{\sqrt{\beta}}{2}\frac{du}{dy},$$
$$\frac{\partial^2 F}{\partial y^2} = \Phi''\left(\frac{\sqrt{\beta}}{2}u\right)\frac{\beta}{4}\left(\frac{\partial u}{\partial y}\right)^2 + \Phi'\left(\frac{\sqrt{\beta}}{2}u\right)\frac{\sqrt{\beta}}{2}\frac{\partial^2 u}{\partial y^2}.$$

Noting that $\Phi''(z) = -z\Phi'(z)$ and dividing by $\Phi'\left(\frac{\sqrt{\beta}}{2}u\right)\frac{\sqrt{\beta}}{2}$ gives

$$\frac{du}{dt} + (t - y^2)\frac{\partial u}{\partial y} + \frac{2}{\beta}\frac{\partial^2 u}{\partial y^2} - \frac{1}{2}u\left(\frac{\partial u}{\partial y}\right)^2 = 0. \tag{8}$$

The $y \to -\infty$ boundary condition on F corresponds to

$$u(t, y) \sim \frac{t - y^2}{\sqrt{|y|}}, \quad y \to -\infty.$$

The boundary condition at $t = +\infty$ is not necessary for the derivation below, but it is discussed in a separate paper by Bloemendal and Sutton [2].

Our goal is to analyze the solution $u(t, y)$ to the above PDE. We shall ultimately arrive at the asymptotic expression (1). We start with the educated guess that the second-derivative term in (8) is negligible as $\beta \to \infty$ and study

$$\frac{du}{dt} + (t - y^2)\frac{\partial u}{\partial y} - \frac{1}{2}u\left(\frac{\partial u}{\partial y}\right)^2 = 0, \tag{9}$$

$$u(t, y) \sim \frac{t - y^2}{\sqrt{-y}}, \quad y \to -\infty \tag{10}$$

Because $F(t, y)$ jumps from 0 to 1 at $y = \frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}$, the transformed $u(t, y)$ should have a sign change along that curve. We intend to verify this and to linearize the solution about the contour $u(t, y) = 0$.

B. Method of characteristics

The first-order PDE (9) can be solved by the method of characteristics. Let $p = \frac{\partial u}{\partial t}$ and $q = \frac{\partial u}{\partial y}$. Then the PDE can be written

$$G(t, y, u, p, q) = p + (t - y^2)q - \frac{1}{2}uq^2 = 0. \tag{11}$$

The solution is a surface in (t, y, u, p, q)-space. We introduce a parameter s and seek a curve $(t(s), y(s), u(s), p(s), q(s))$ along the surface. This is obtained from an initial condition $(t_0, y_0, u_0, p_0, q_0)$ and the characteristic strip equations [19]

$$\frac{\partial t}{\partial s} = G_p = 1,$$
$$\frac{\partial y}{\partial s} = G_q = (t - y^2) - uq,$$
$$\frac{\partial u}{\partial s} = pG_p + qG_q = p + (t - y^2)q - uq^2,$$
$$\frac{\partial p}{\partial s} = -G_t - pG_u = -q + \frac{1}{2}pq^2,$$
$$\frac{\partial q}{\partial s} = -G_y - qG_u = 2yq + \frac{1}{2}q^3.$$

Note that $G = 0$ implies $p = -(t - y^2)q + \frac{1}{2}uq^2$. Hence, p can be eliminated:

$$\frac{\partial t}{\partial s} = 1,$$
$$\frac{\partial y}{\partial s} = (t - y^2) - uq,$$
$$\frac{\partial u}{\partial s} = -\frac{1}{2}uq^2,$$
$$\frac{\partial q}{\partial s} = 2yq + \frac{1}{2}q^3.$$

Recall that we are most interested in the region around $u \approx 0$. Perhaps there is a solution to the characteristic strip equations with $u(s) = 0$ identically. Taking $t_0 = s_0$, this would require $t(s) = s$ and $\frac{\partial y}{\partial s} = (s - y^2)$, which implies $y(s) = \frac{\mathrm{Ai}'(s)}{\mathrm{Ai}(s)}$. This reduces the differential equation for q to

$$\frac{\partial q}{\partial s} = 2\frac{\mathrm{Ai}'(s)}{\mathrm{Ai}(s)}q + \frac{1}{2}q^3.$$

Every function of the form

$$q(s) = \pm\frac{\mathrm{Ai}(s)^2}{\sqrt{\int_s^b \mathrm{Ai}(w)^4 dw}} \tag{12}$$

is a solution. The initial condition is determined by (10). We find $p \sim \frac{1}{\sqrt{-y}} \sim \frac{1}{\sqrt{-\mathrm{Ai}'(s)/\mathrm{Ai}(s)}} \sim s^{-1/4}$ as $s \to +\infty$. So in (11), $p \sim s^{-1/4}$, $t - y^2 \sim -\frac{1}{2}s^{-1/2}$, and therefore $q \sim 2s^{1/4}$ as $s \to +\infty$. This fixes a positive sign on (12). Further, if the upper limit of integration b were less than $+\infty$, then (12) would decay exponentially. However, with $b = +\infty$, the solution has precisely the desired asymptotics $q(s) \sim 2s^{1/4}$ as $s \to +\infty$.

Solving for p, the characteristic is

$$\begin{aligned}
t &= s,\\
y &= \frac{\mathrm{Ai}'(s)}{\mathrm{Ai}(s)},\\
u &= 0,\\
p &= \frac{-s\,\mathrm{Ai}(s)^2 + \mathrm{Ai}'(s)^2}{\sqrt{\int_s^\infty \mathrm{Ai}(w)^4 dw}},\\
q &= \frac{\mathrm{Ai}(s)^2}{\sqrt{\int_s^\infty \mathrm{Ai}(w)^4 dw}}.
\end{aligned}$$

C. Eigenvalue asymptotics

The Tracy-Widom distribution focuses attention on $y = +\infty$, i.e., $s = a_1$. At this terminal point of the characteristic,

$$p(a_1) = \frac{-a_1\,\mathrm{Ai}(a_1)^2 + \mathrm{Ai}'(a_1)^2}{\sqrt{\int_{a_1}^\infty \mathrm{Ai}(w)^4 dw}} = \left(\int_0^\infty \left(\tfrac{1}{\mathrm{Ai}'(a_1)}\,\mathrm{Ai}(x + a_1)\right)^4 dx\right)^{-1/2}.$$

Hence, $u(t, +\infty)$ can be linearized as follows:

$$u(t, +\infty) \sim 0 + p(a_1)(t - a_1), \quad t \to a_1.$$

Then, the Tracy-Widom distribution is approximated by

$$\boxed{F(t, +\infty) \sim \Phi\left(\frac{\sqrt{\beta}}{2} p(a_1)(t - a_1)\right), \quad t \to a_1, \ \beta \to \infty.}$$

That is, for large β, the distribution should be approximately normal with mean a_1 and standard deviation

$$\frac{2}{\sqrt{\beta}} \frac{1}{p(a_1)} = \frac{2}{\sqrt{\beta}} \sqrt{\int_0^\infty \left(\tfrac{1}{\mathrm{Ai}'(a_1)} \mathrm{Ai}(x + a_k)\right)^4 dx}.$$

This is another argument supporting (1).

V. CONCLUSION

General-β random matrix theory is still a challenging business. We have shown how the asymptotic regime $\beta \to \infty$ can suggest new methods and provide new data.

---

[1] James T. Albrecht, Cy P. Chan, and Alan Edelman. Sturm sequences and random eigenvalue distributions. *Foundations of Computational Mathematics*, 9(4):461–483, 2009.

[2] Alex Bloemendal and Brian D. Sutton. General-beta computation at the soft edge. In preparation.

[3] Alex Bloemendal and Bálint Virág. Limits of spiked random matrices I. *arXiv:1011.1877*, November 2010.

[4] Alex Bloemendal and Bálint Virág. Limits of spiked random matrices II. *arXiv:1109.3704*, September 2011.

[5] F. Bornemann. On the numerical evaluation of distributions in random matrix theory: a review. *Markov Processes and Related Fields*, 16(4):803–866, 2010.

[6] R. Courant and D. Hilbert. *Methods of mathematical physics. Vol. I.* Interscience Publishers, Inc., New York, N.Y., 1953.

[7] NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.0.6 of 2013-05-06. Online companion to [14].

[8] Ioana Dumitriu and Alan Edelman. Matrix models for beta ensembles. *J. Math. Phys.*, 43(11):5830–5847, 2002.

[9] Ioana Dumitriu and Alan Edelman. Eigenvalues of Hermite and Laguerre ensembles: large beta asymptotics. *Annales de l'Institut Henri Poincaré. Probabilités et Statistiques*, 41(6):1083–1099, 2005.

[10] Freeman J. Dyson. Statistical theory of the energy levels of complex systems. I. *Journal of Mathematical Physics*, 3:140–156, 1962.

[11] Alan Edelman. Stochastic differential equations and random matrices. SIAM Conference on Applied Linear Algebra, Williamsburg, Virginia, 2003. http://www-math.mit.edu/%7Eedelman/homepage/talks/siam2003.ppt.

[12] Alan Edelman and Brian D. Sutton. From random matrices to stochastic operators. *J. Stat. Phys.*, 127(6):1121–1165, 2007.

[13] Tosio Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[14] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST Handbook of Mathematical Functions*. Cambridge University Press, New York, NY, 2010. Print companion to [7].

[15] José Ramírez, Brian Rider, and Bálint Virág. Beta ensembles, stochastic Airy spectrum, and a diffusion. *J. Amer. Math. Soc.*, 24(4):919–944, 2011.

[16] Brian D. Sutton. *The stochastic operator approach to random matrix theory*. PhD thesis, Massachusetts Institute of Technology, Cambridge, MA, 2005.

[17] Craig A. Tracy and Harold Widom. Level-spacing distributions and the Airy kernel. *Physics Letters. B*, 305(1-2):115–118, 1993.

[18] Craig A. Tracy and Harold Widom. On orthogonal and symplectic matrix ensembles. *Communications in Mathematical Physics*, 177(3):727–754, 1996.

[19] Daniel Zwillinger. *Handbook of differential equations*. Academic Press Inc., Boston, MA, 1989.