

Mesh Generation for Implicit Geometries

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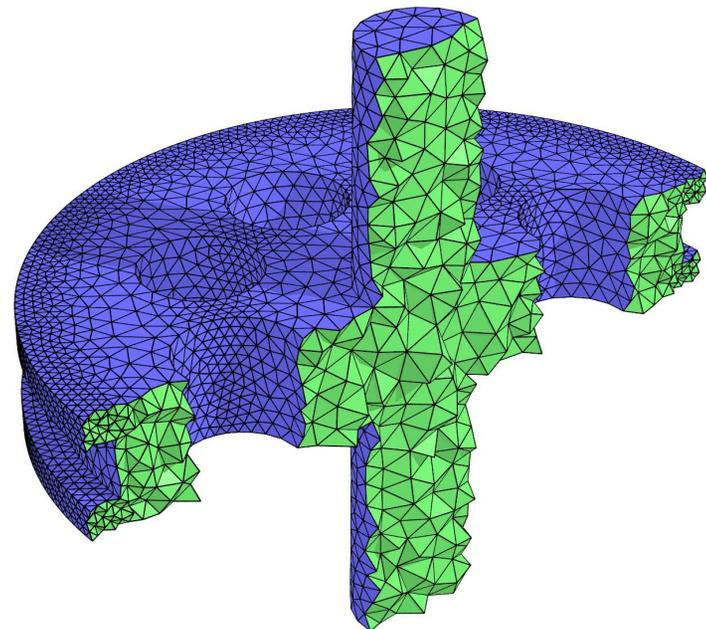
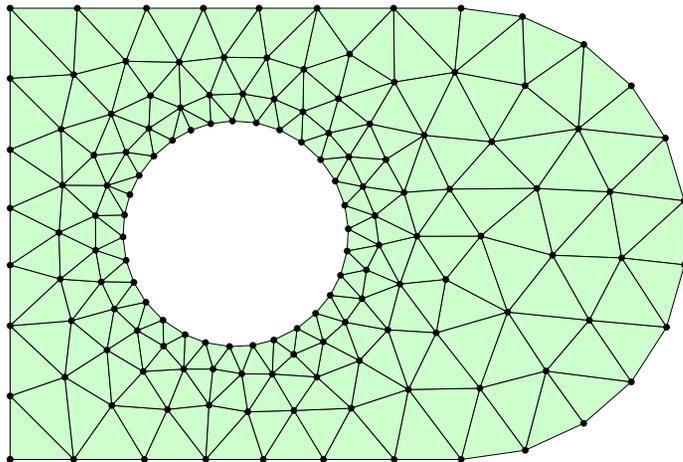
Advisors: Alan Edelman and Gilbert Strang

Topics

1. **Introduction**
2. The New Mesh Generator
3. Mesh Size Functions
4. Applications
5. Conclusions

Mesh Generation

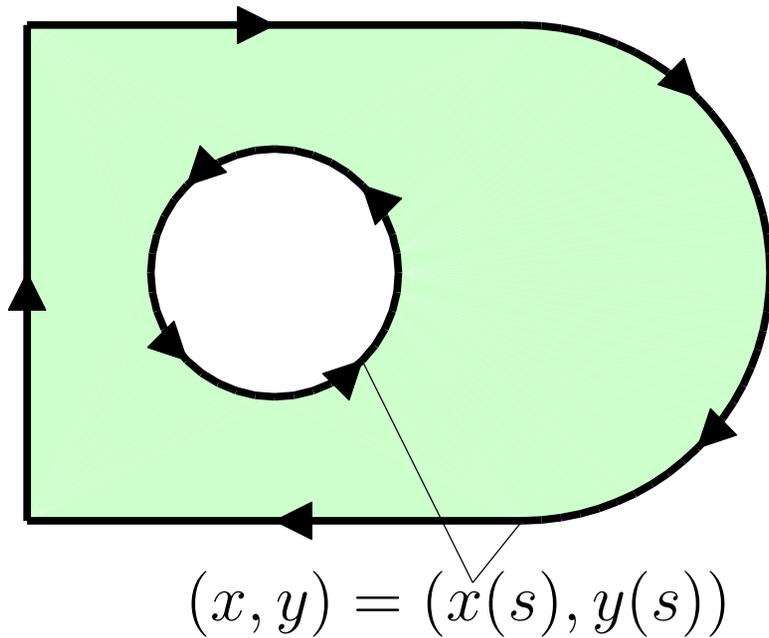
- Given a geometry, determine node points and element connectivity
- Resolve the geometry and high element qualities, but few elements
- Applications: Numerical solution of PDEs (FEM, FVM, BEM), interpolation, computer graphics, visualization
- Popular algorithms: Delaunay refinement, Advancing front, Octree



Geometry Representations

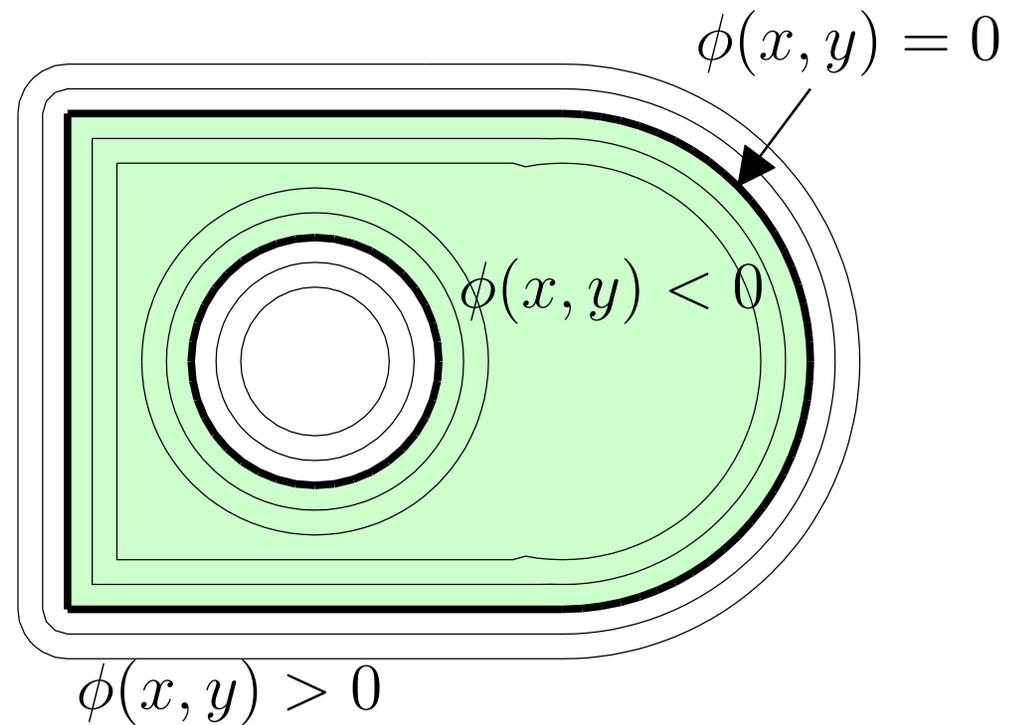
Explicit Geometry

- Parameterized boundaries



Implicit Geometry

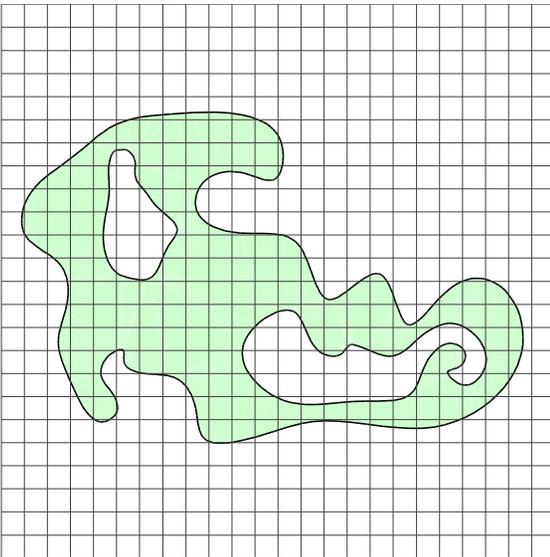
- Boundaries given by zero level set



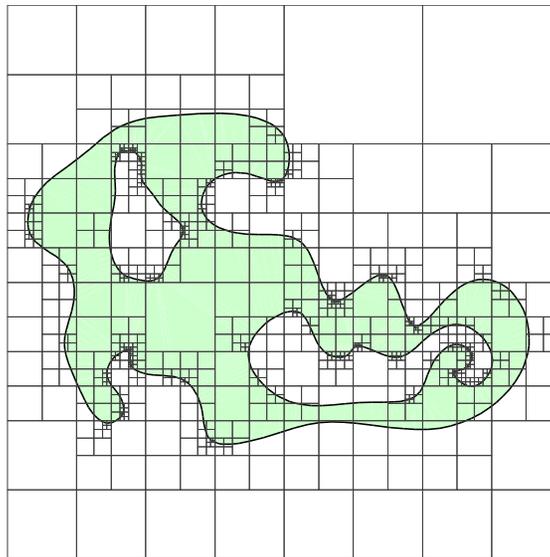
Discretized Implicit Geometries

- Discretize implicit function ϕ on *background grid*
 - Cartesian, Octree, or unstructured
- Obtain $\phi(\boldsymbol{x})$ for general \boldsymbol{x} by interpolation

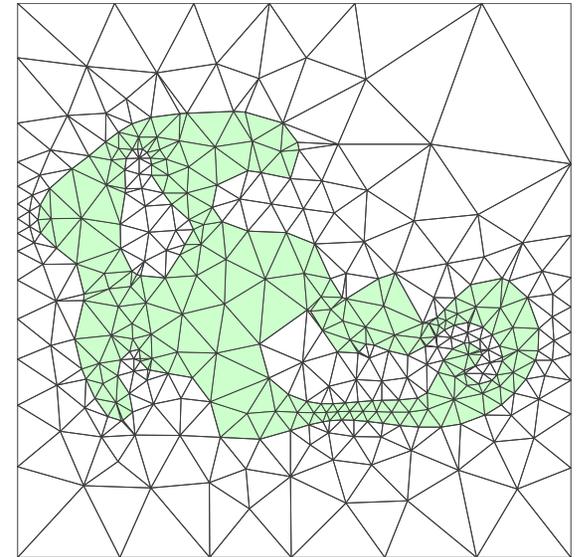
Cartesian



Quadtree/Octree



Unstructured



Explicit vs Implicit

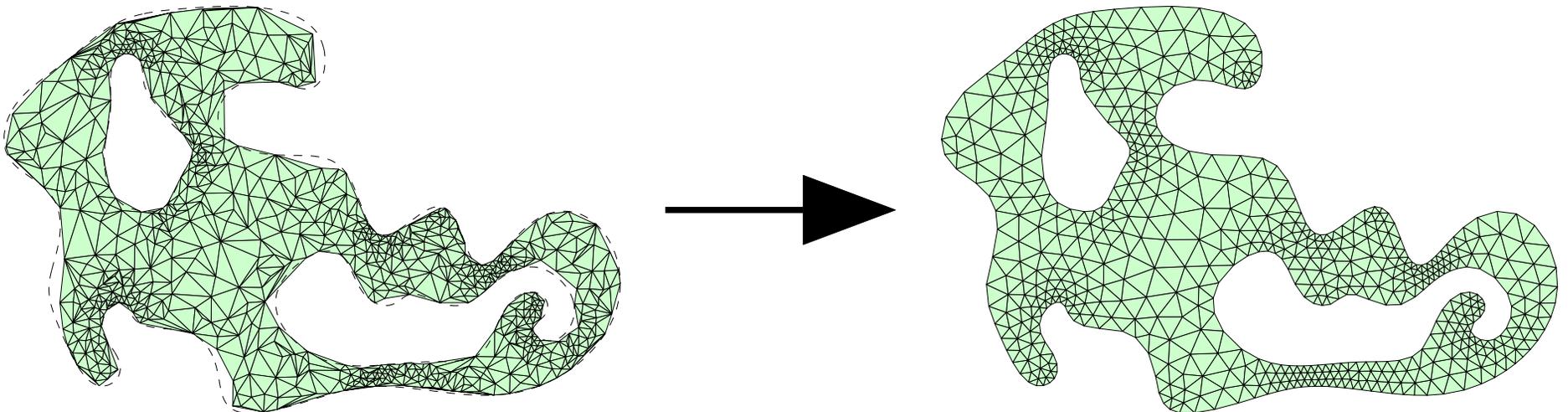
- Most CAD programs represent geometries explicitly (NURBS and B-rep)
- Most mesh generators require explicit expressions for boundaries (Delaunay refinement, Advancing front)
- Increasing interest for implicit geometries:
 - Easy to implement: Set operations, offsets, blendings, etc
 - Extend naturally to higher dimensions
 - Image based geometry representations (MRI/CT scans)
 - Level set methods for evolving interfaces

Topics

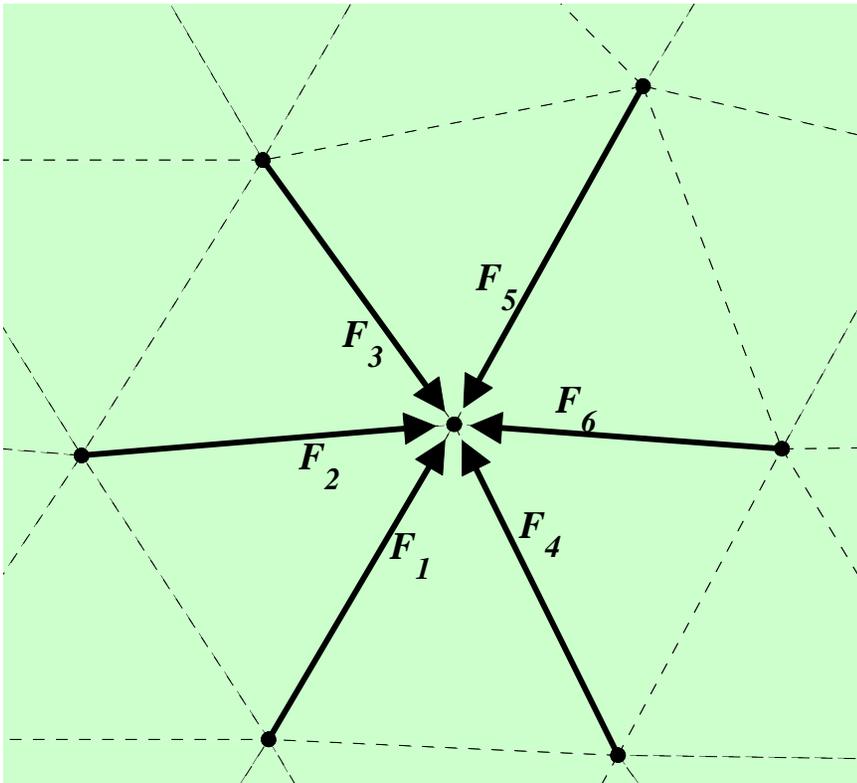
1. Introduction
2. **The New Mesh Generator**
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The New Mesh Generator

1. Start with *any* topologically correct initial mesh, for example random node distribution and Delaunay triangulation
2. Move nodes to find force equilibrium in edges
 - Project boundary nodes using ϕ
 - Update element connectivities



Internal Forces



For each *interior* node:

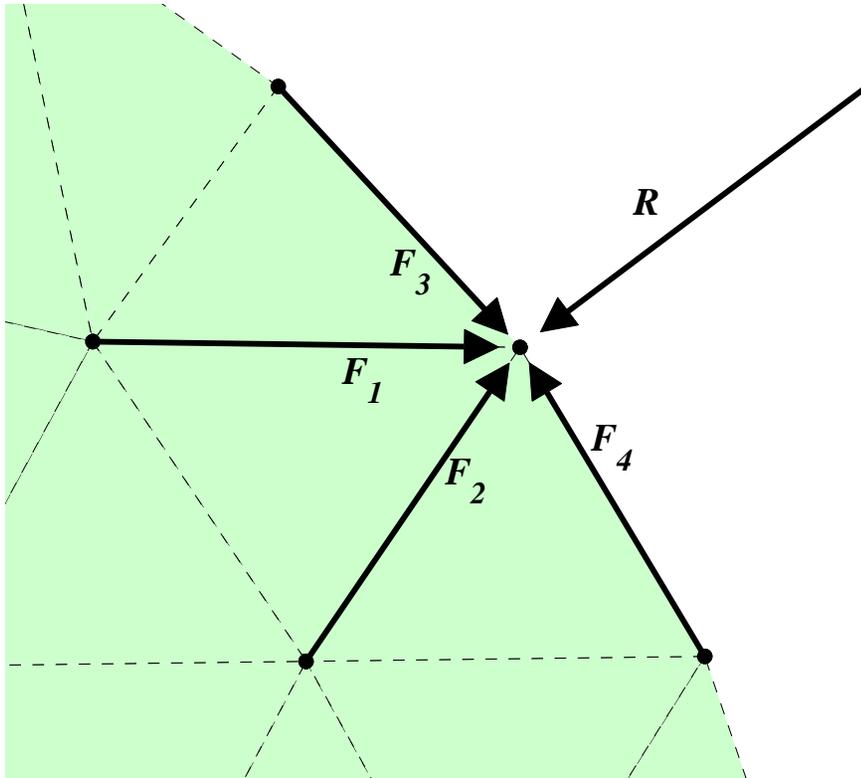
$$\sum_i \mathbf{F}_i = 0$$

Repulsive forces depending on edge length ℓ and equilibrium length ℓ_0 :

$$|\mathbf{F}| = \begin{cases} k(\ell_0 - \ell) & \text{if } \ell < \ell_0, \\ 0 & \text{if } \ell \geq \ell_0. \end{cases}$$

Get expanding mesh by choosing ℓ_0 larger than desired length h

Reactions at Boundaries



For each *boundary* node:

$$\sum_i F_i + R = 0$$

Reaction force R :

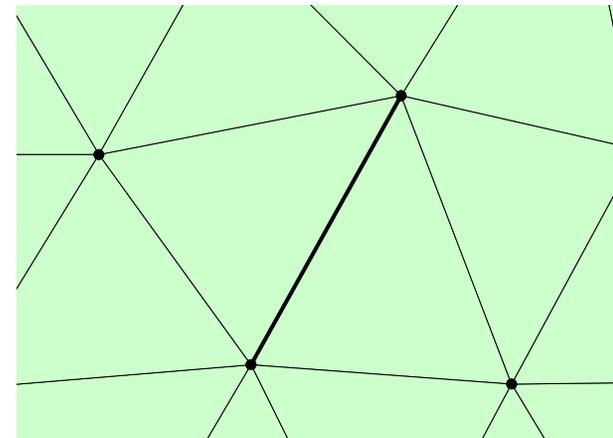
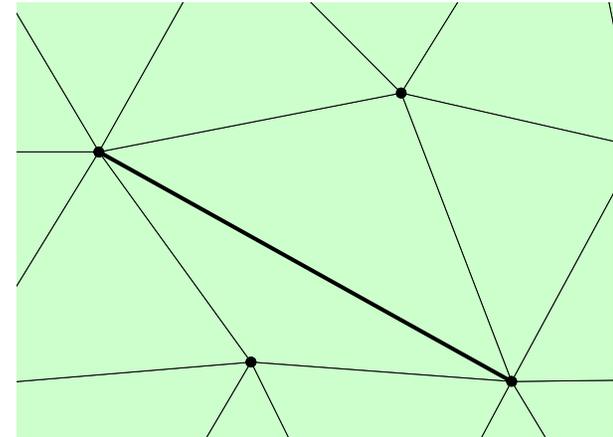
- Orthogonal to boundary
- Keeps node along boundary

Node Movement and Connectivity Updates

- Move nodes \mathbf{p} to find force equilibrium:

$$\mathbf{p}_{n+1} = \mathbf{p}_n + \Delta t \mathbf{F}(\mathbf{p}_n)$$

- Project boundary nodes to $\phi(\mathbf{p}) = 0$
- Elements deform, change connectivity based on element quality or in-circle condition (Delaunay)



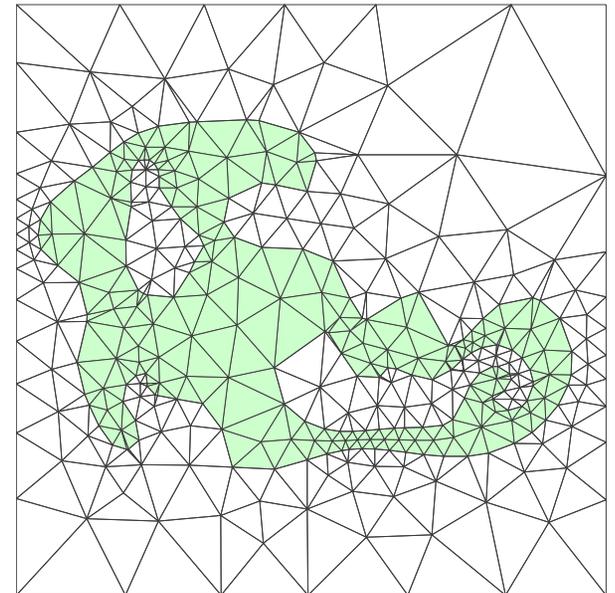
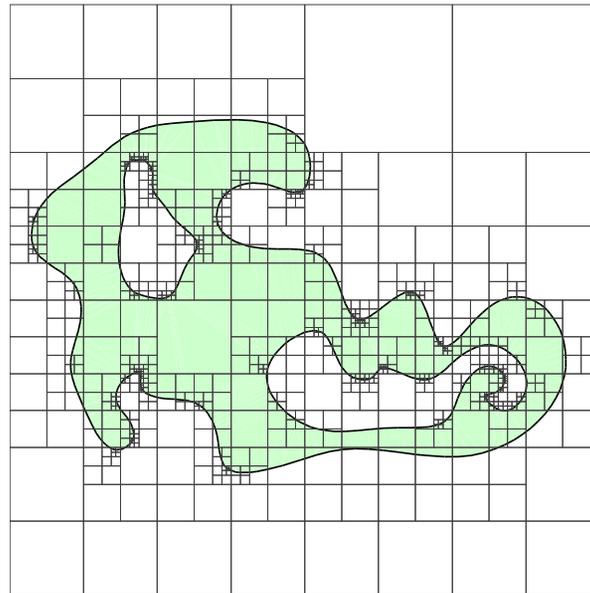
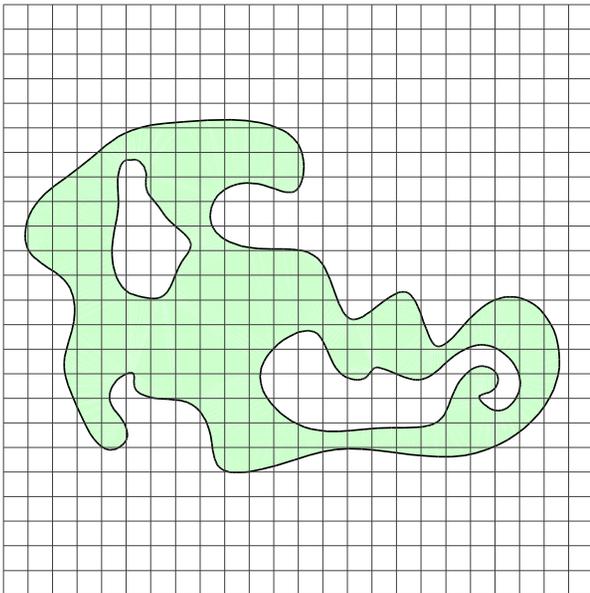
MATLAB Demo

Topics

1. Introduction
2. The New Mesh Generator
3. **Mesh Size Functions**
4. Applications
5. Conclusions

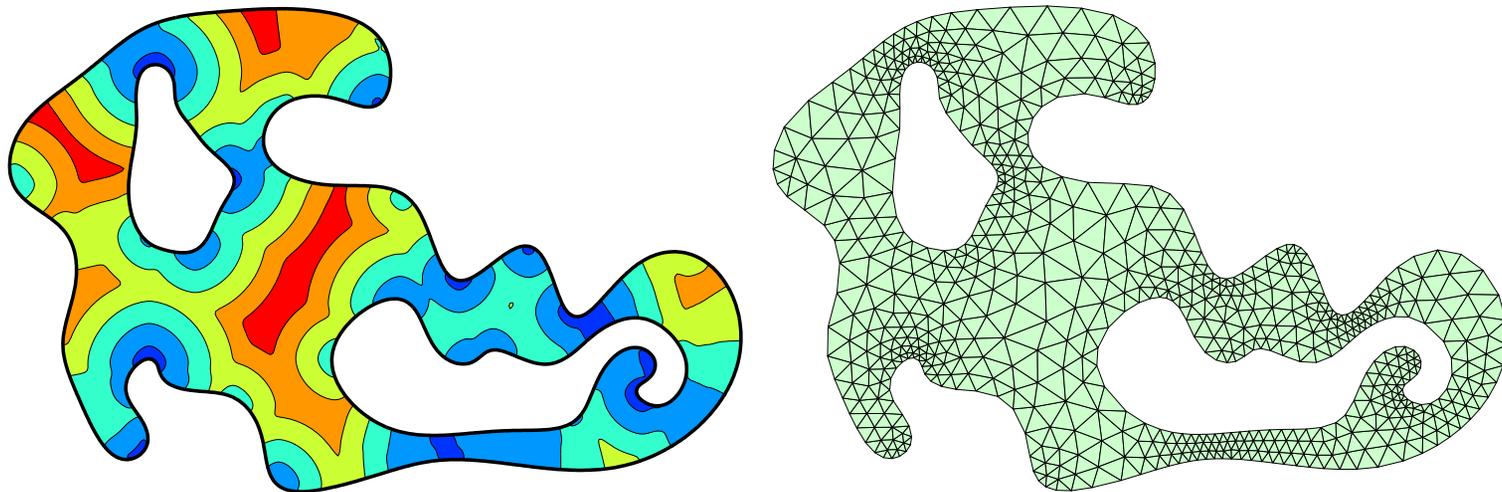
Mesh Size Functions

- Function $h(\boldsymbol{x})$ specifying desired mesh element size
- Many mesh generators need a priori mesh size functions
 - Our force-based meshing
 - Advancing front and Paving methods
- Discretize mesh size function $h(\boldsymbol{x})$ on a coarse background grid



Mesh Size Functions

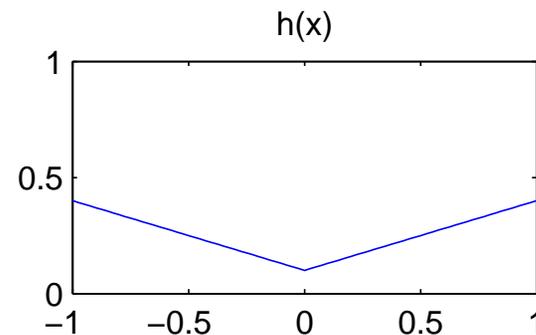
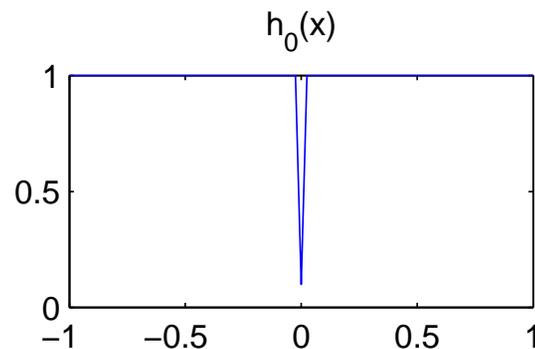
- Based on several factors:
 - Curvature of geometry boundary
 - Local feature size of geometry
 - Numerical error estimates (adaptive solvers)
 - Any user-specified size constraints
- Also: $|\nabla h(\mathbf{x})| \leq g$ to limit ratio $G = g + 1$ of neighboring element sizes



Mesh Gradation

- For a given size function $h_0(\mathbf{x})$, find *gradient limited* $h(\mathbf{x})$ satisfying
 - $|\nabla h(\mathbf{x})| \leq g$
 - $h(\mathbf{x}) \leq h_0(\mathbf{x})$
 - $h(\mathbf{x})$ as large as possible
- Example: A *point-source* in \mathbb{R}^n

$$h_0(\mathbf{x}) = \begin{cases} h_{\text{pnt}}, & \mathbf{x} = \mathbf{x}_0 \\ \infty, & \text{otherwise} \end{cases} \Rightarrow h(\mathbf{x}) = h_{\text{pnt}} + g|\mathbf{x} - \mathbf{x}_0|$$



Mesh Gradation

- Example: Any shape, with distance function $d(\mathbf{x})$, in \mathbb{R}^n

$$h_0(\mathbf{x}) = \begin{cases} h_{\text{shape}}, & d(\mathbf{x}) = 0 \\ \infty, & \text{otherwise} \end{cases} \Rightarrow h(\mathbf{x}) = h_{\text{shape}} + gd(\mathbf{x})$$

- Example: Any function $h_0(\mathbf{x})$ in \mathbb{R}^n

$$h(\mathbf{x}) = \min_{\mathbf{y}} (h_0(\mathbf{y}) + g|\mathbf{x} - \mathbf{y}|)$$

Too expensive and inflexible for practical use

The Gradient Limiting Equation

- The optimal *gradient limited* $h(\mathbf{x})$ is the steady-state solution to

$$\frac{\partial h}{\partial t} + |\nabla h| = \min(|\nabla h|, g),$$
$$h(t = 0, \mathbf{x}) = h_0(\mathbf{x}).$$

- Nonlinear hyperbolic PDE (a Hamilton-Jacobi equation)
- **Theorem:** For g constant, and convex domain, the steady-state solution is

$$h(\mathbf{x}) = \min_{\mathbf{y}} (h_0(\mathbf{y}) + g|\mathbf{x} - \mathbf{y}|)$$

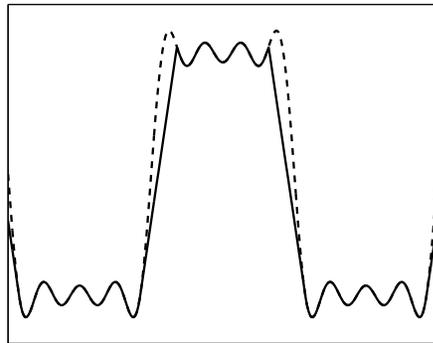
Proof. Use the Hopf-Lax theorem (see paper).

- The optimal minimum-over-point-sources solution!

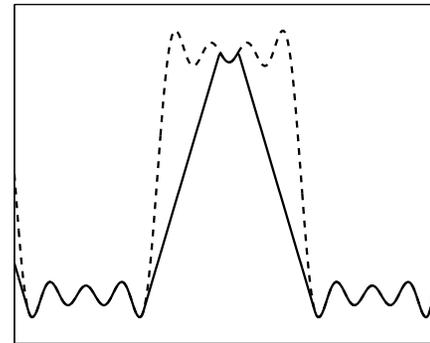
Gradient Limiting in 1-D

- Dashed = Original function, Solid = Gradient limited function
- Very different from smoothing!

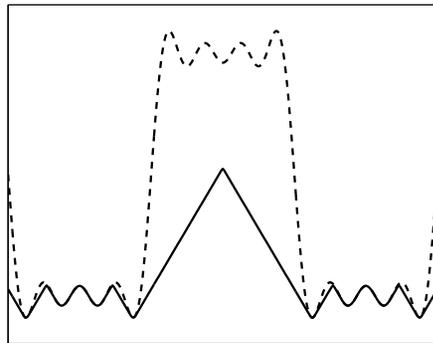
Max Gradient $g = 4$



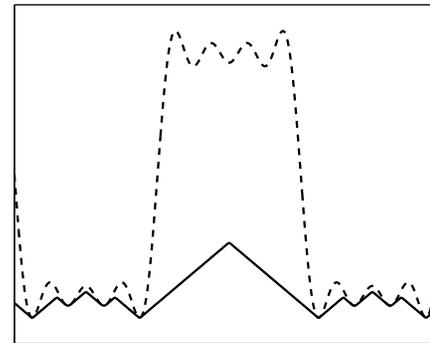
Max Gradient $g = 2$



Max Gradient $g = 1$



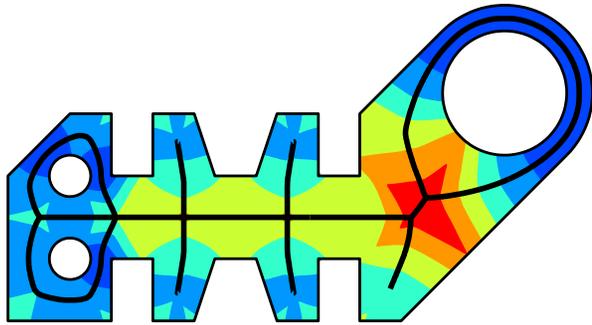
Max Gradient $g = 0.5$



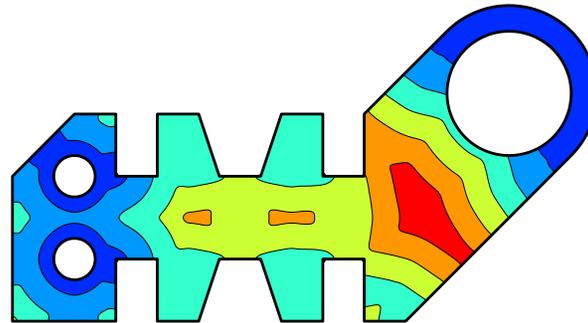
Mesh Size Functions – 2-D Examples

- Feature size computed by $h_{\text{ifs}}(\mathbf{x}) = |\phi(\mathbf{x})| + |\phi_{\text{MA}}(\mathbf{x})|$

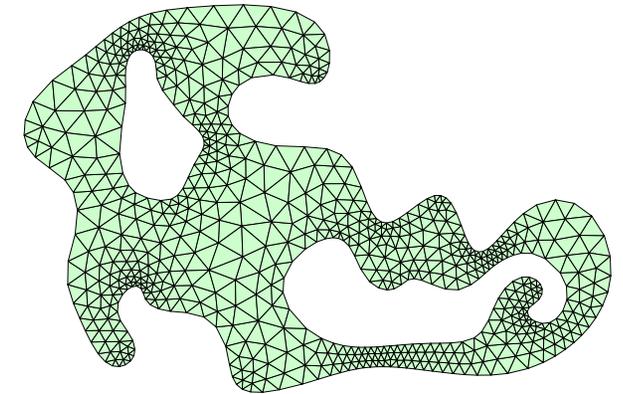
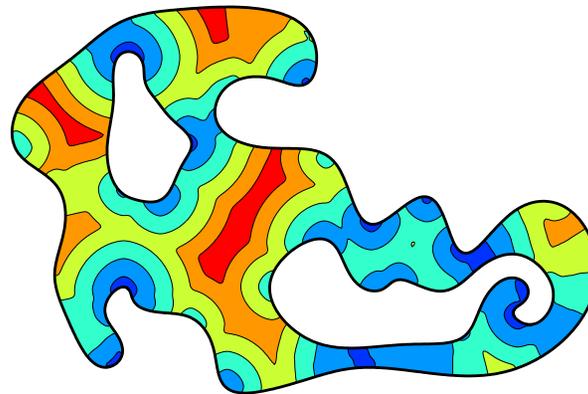
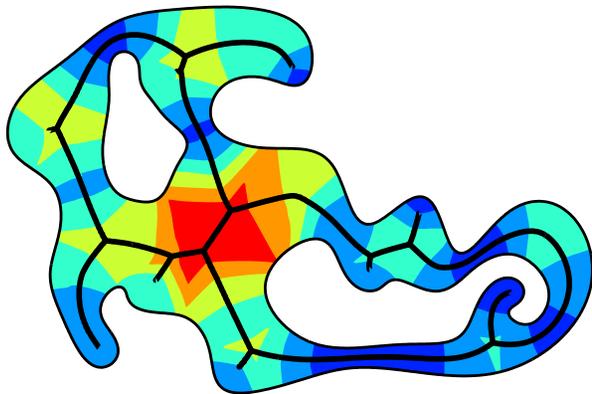
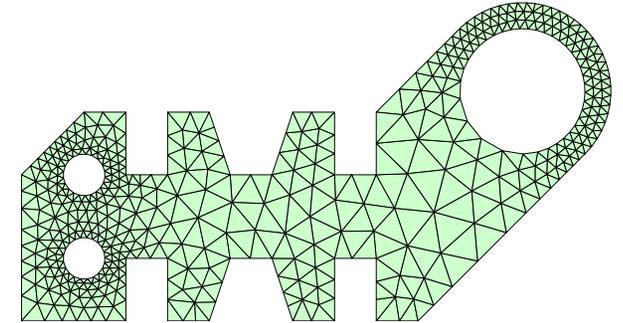
Med. Axis & Feature Size



Mesh Size Function $h(x)$

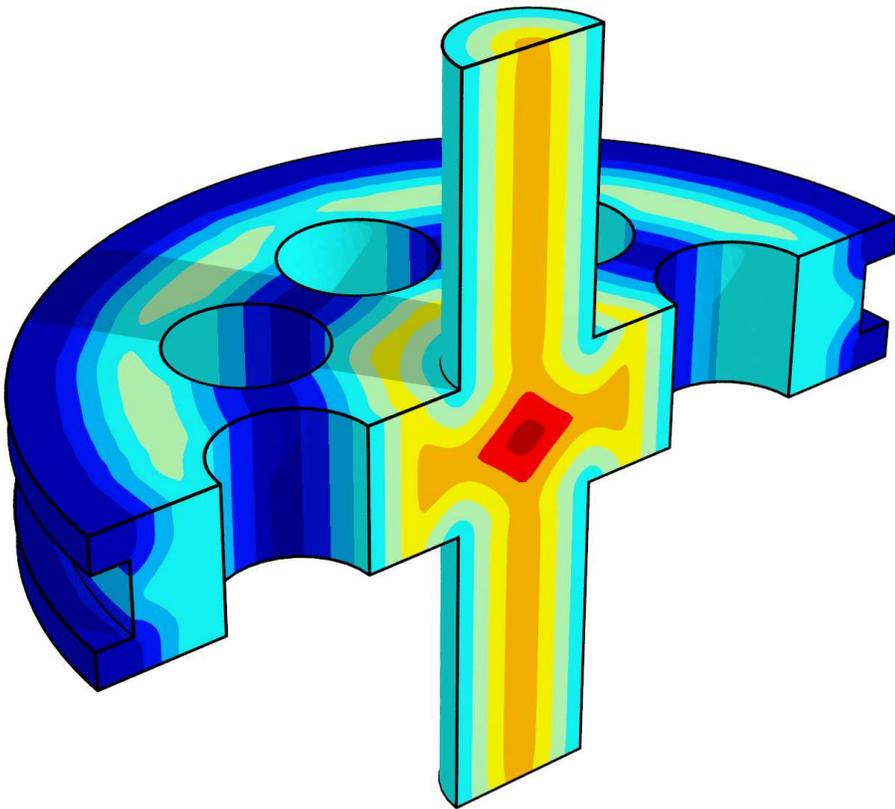


Mesh Based on $h(x)$

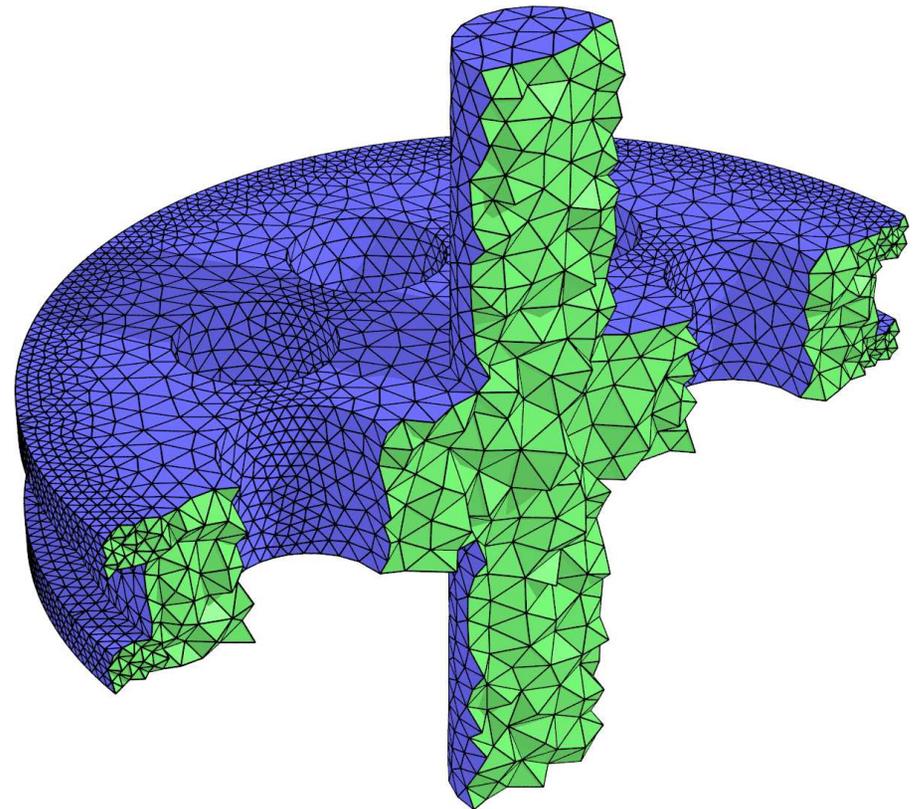


Mesh Size Functions – 3-D Examples

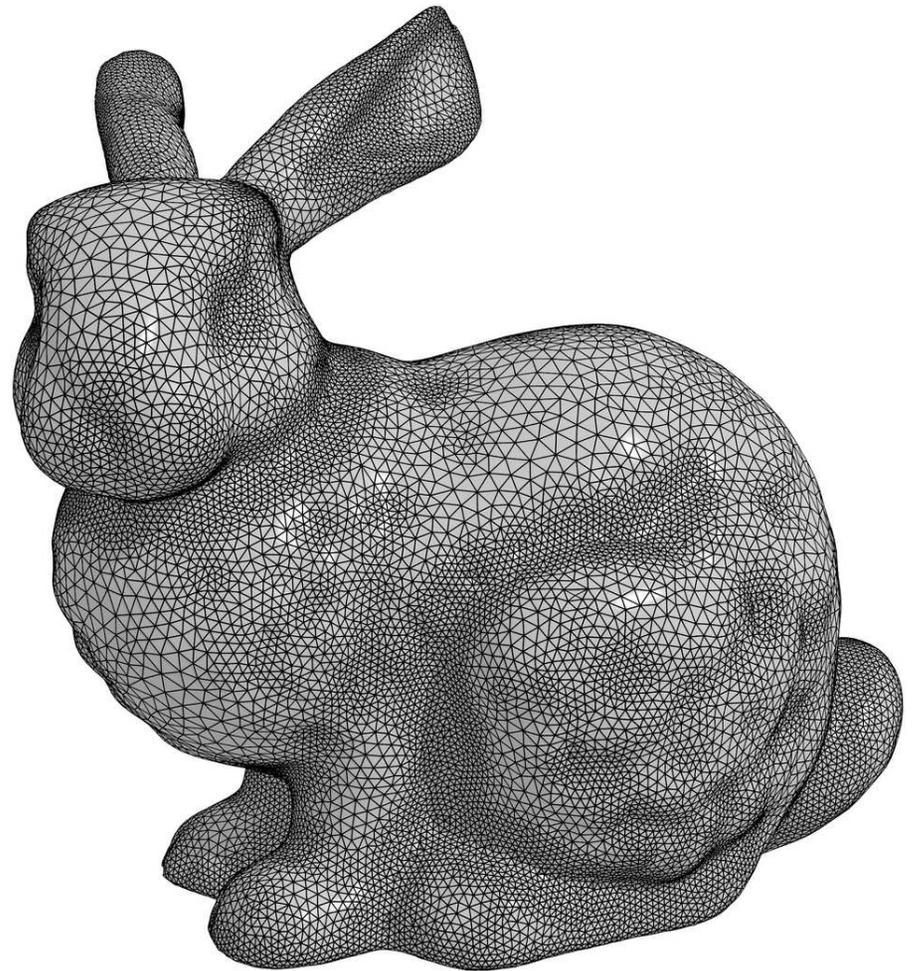
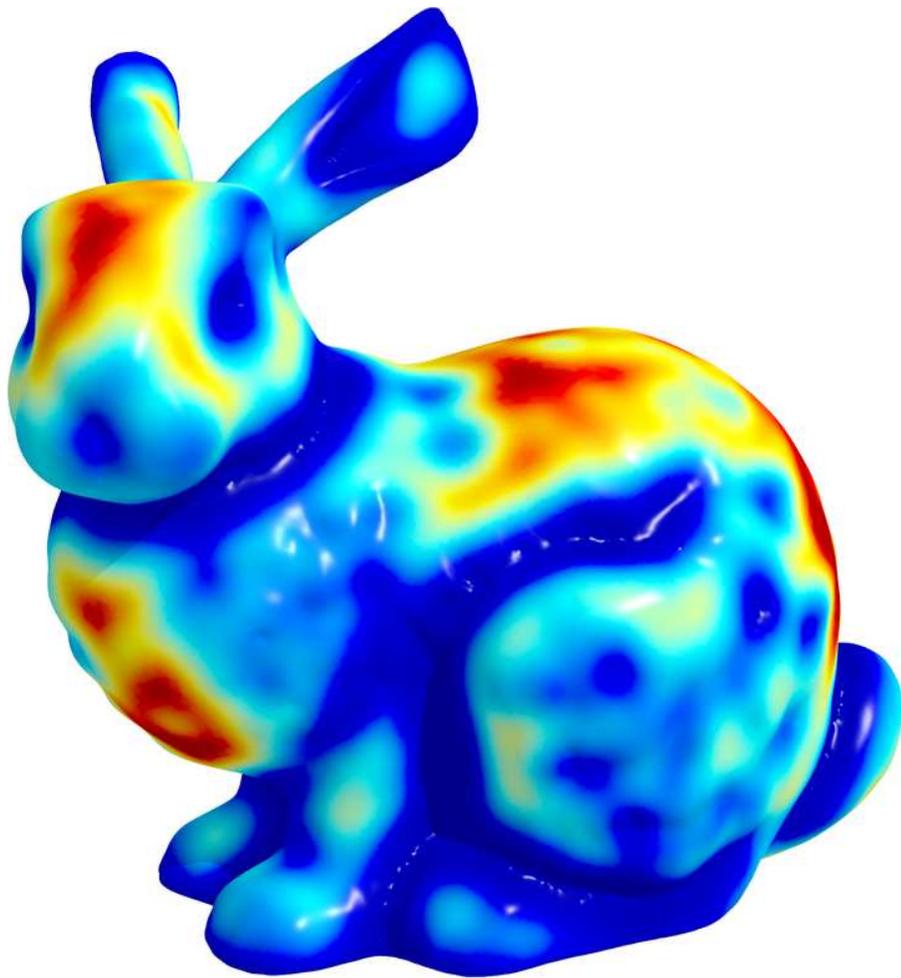
Mesh Size Function $h(x)$



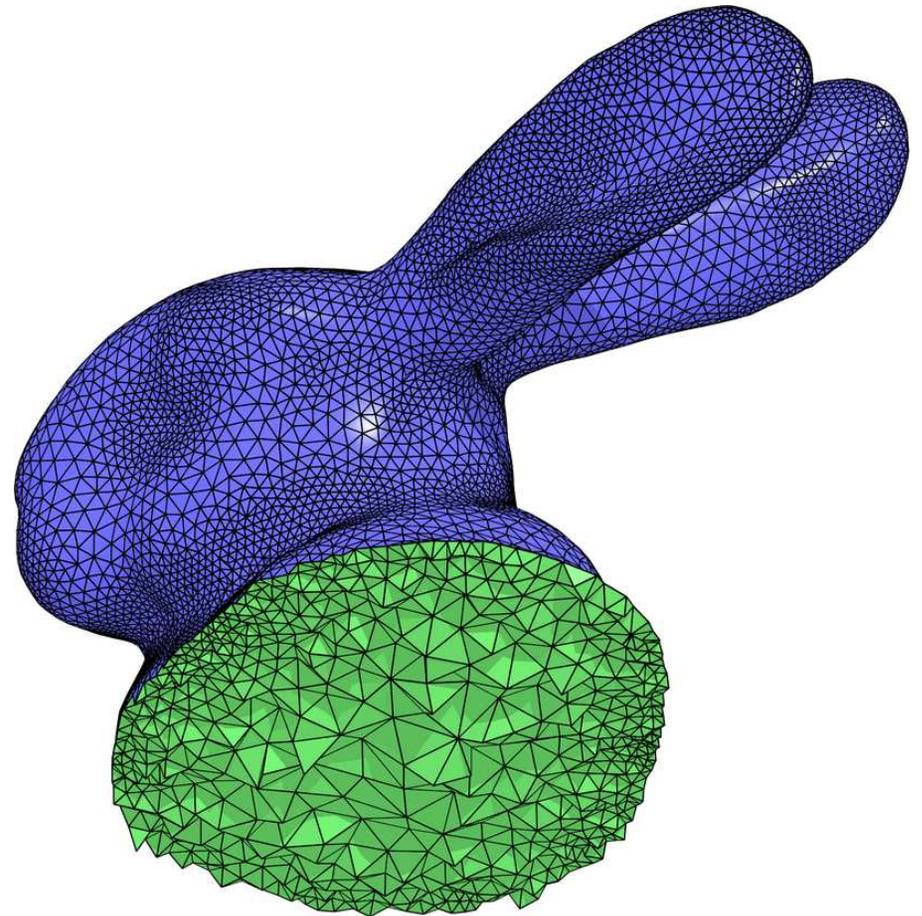
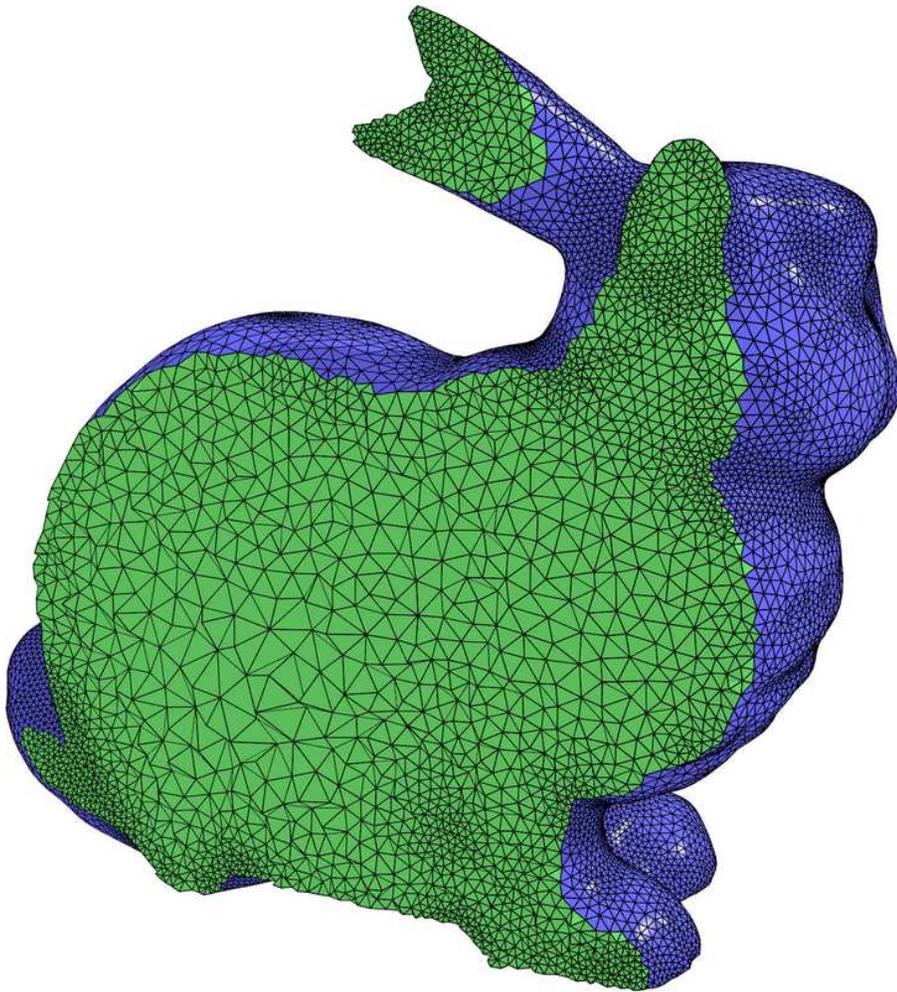
Mesh Based on $h(x)$



Mesh Size Functions – 3-D Examples



Mesh Size Functions – 3-D Examples

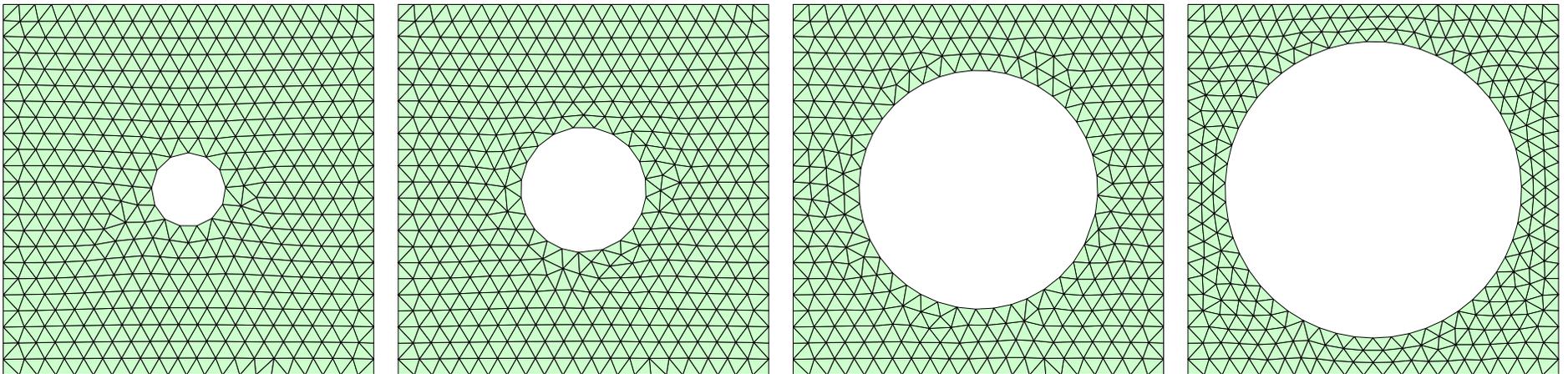


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Moving Meshes

- Iterative formulation well-suited for geometries with moving boundaries
- Mesh from previous time step good initial condition, a new mesh obtained by a few additional iterations
- Even better if smooth embedding of boundary velocity – move all nodes
- Density control if area changes



Level Sets and Finite Elements

- Level Set Methods superior for interface propagation:
 - Numerical stability and entropy satisfying solutions
 - Topology changes easily handled, in particular in three dimensions
- Unstructured meshes better for the physical problems:
 - Better handling of boundary conditions
 - Easy mesh grading and adaptation
- Proposal: Combine Level Sets and Finite Elements
 - Use level sets on background grid for interface
 - Mesh using our iterations, and solve physical problem using FEM
 - Interpolate boundary velocity to background grid

Structural Vibration Control

- Consider eigenvalue problem

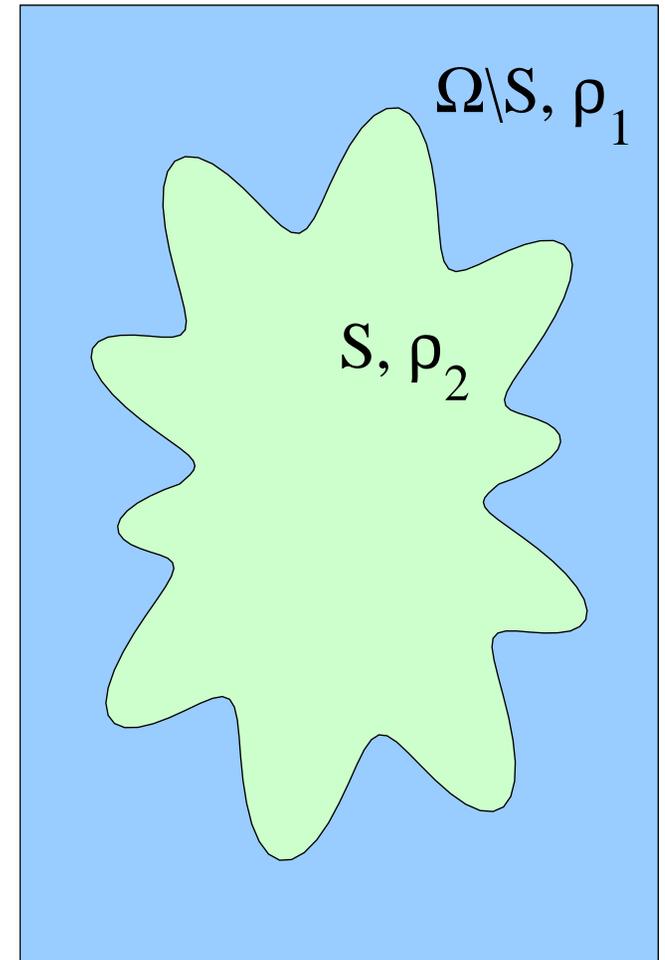
$$\begin{aligned} -\Delta u &= \lambda \rho(\mathbf{x}) u, & x \in \Omega \\ u &= 0, & x \in \partial\Omega. \end{aligned}$$

with

$$\rho(\mathbf{x}) = \begin{cases} \rho_1 & \text{for } x \notin S \\ \rho_2 & \text{for } x \in S. \end{cases}$$

- Solve the optimization

$$\min_S \lambda_1 \text{ or } \lambda_2 \text{ subject to } \|S\| = K.$$



Structural Vibration Control

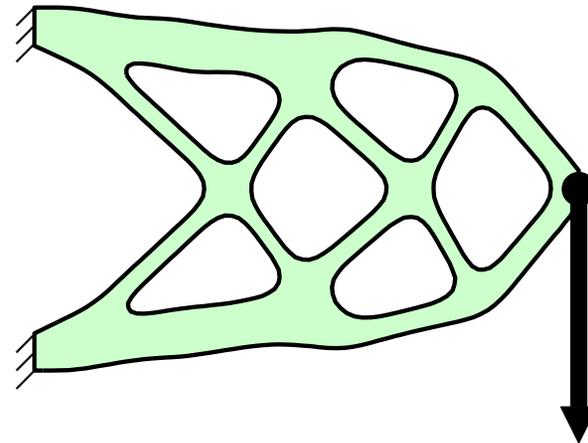
- Level set formulation by Osher and Santosa:
 - Calculate descent direction $\delta\phi = -v(\mathbf{x})|\nabla\phi|$ using solution λ_i, u_i
 - Find Lagrange multiplier for area constraint using Newton
 - Represent interface implicitly, propagate using level set method
- Use finite elements on unstructured mesh for eigenvalue problem
 - More accurate interface condition
 - Arbitrary geometries
 - Graded meshes

Structural Design Improvement

- Linear elastostatics, minimize compliance

$$\int_{\partial\Omega} g \cdot u \, ds = \int_{\Omega} A\epsilon(u) \cdot \epsilon(u) \, dx \text{ subject to } \|\Omega\| = K.$$

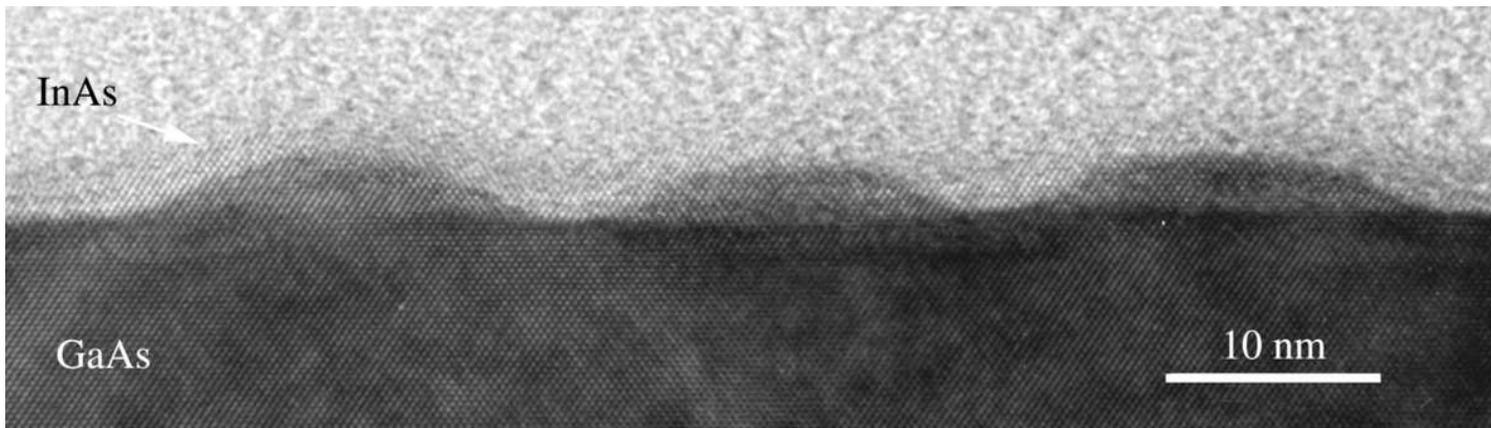
- Boundary variation methods by Murat and Simon, Pironneau, Homogenization method by Bendsoe and Kikuchi
- Level set approaches by Sethian and Wiegmann (Immersed interface method) and Allaire et al (*Ersatz material*, Young's modulus ε)



Stress Driven Rearrangement Instabilities

- Epitaxial growth of InAs on a GaAs substrate
- Stress from misfit in lattice parameters
- Quasi-static interface evolution, descent direction for elastic energy and surface energy

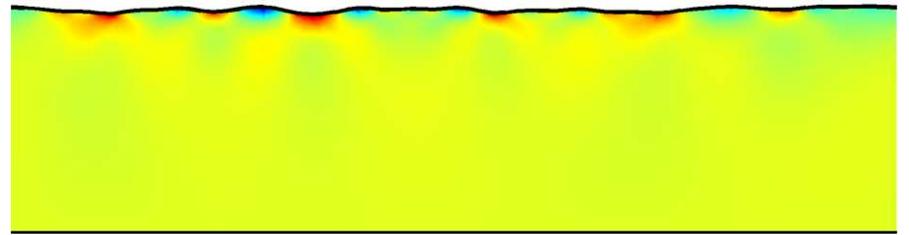
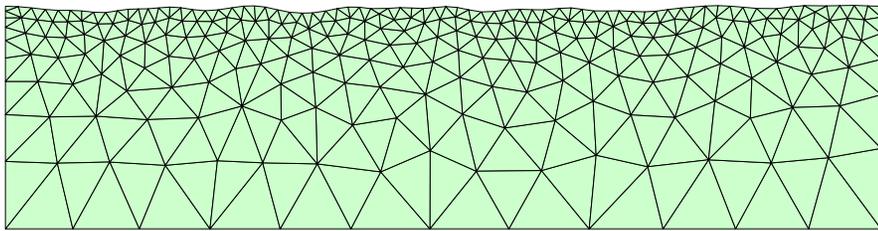
$$\frac{\partial \phi}{\partial \tau} + F(\mathbf{x}) |\nabla \phi| = 0, \text{ with } F(\mathbf{x}) = \varepsilon(\mathbf{x}) - \sigma \kappa(\mathbf{x})$$



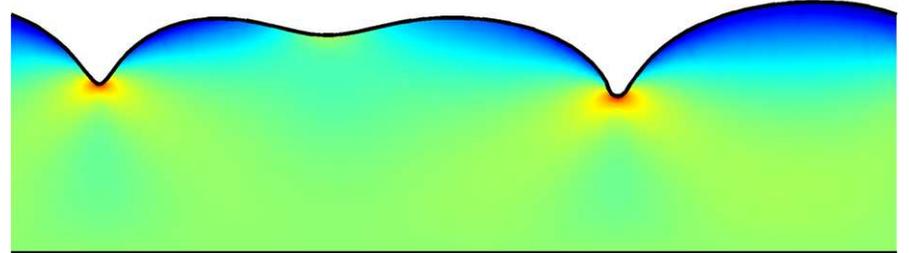
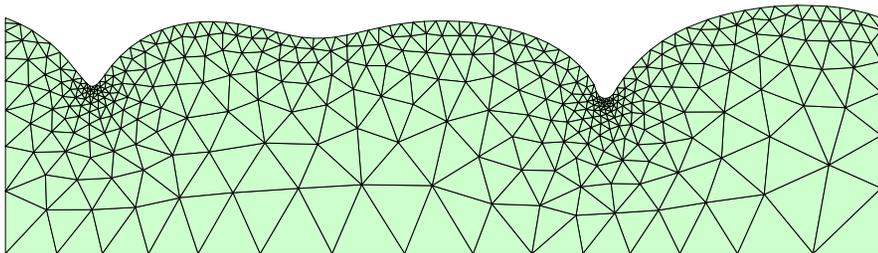
Electron micrograph of defect-free InAs quantum dots

Stress Driven Rearrangement Instabilities

Initial Configuration

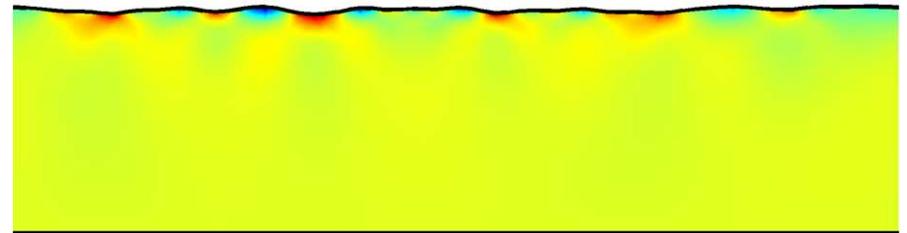
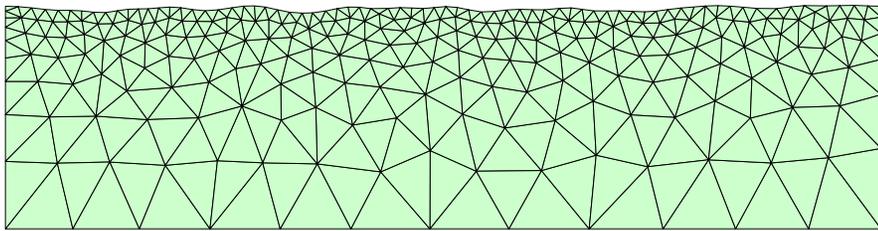


Final Configuration, $\sigma = 0.20$

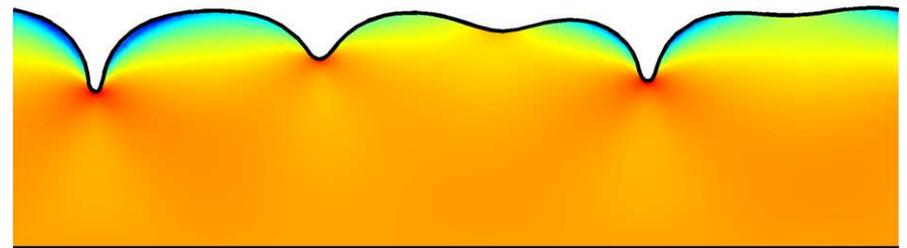
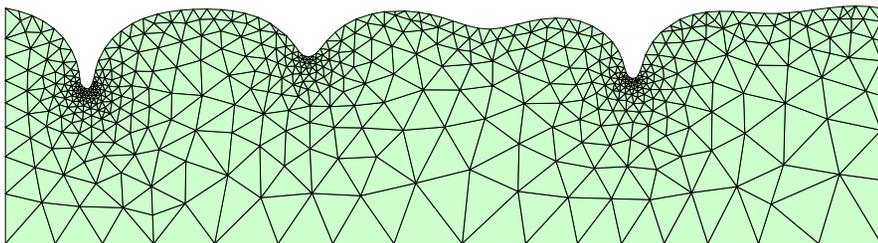


Stress Driven Rearrangement Instabilities

Initial Configuration

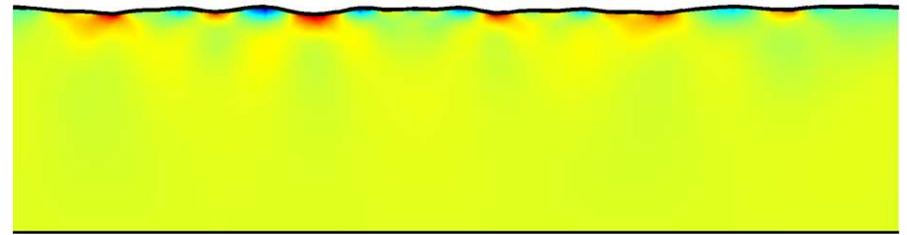
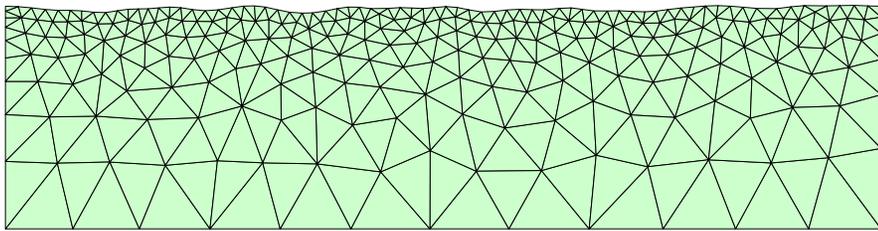


Final Configuration, $\sigma = 0.10$

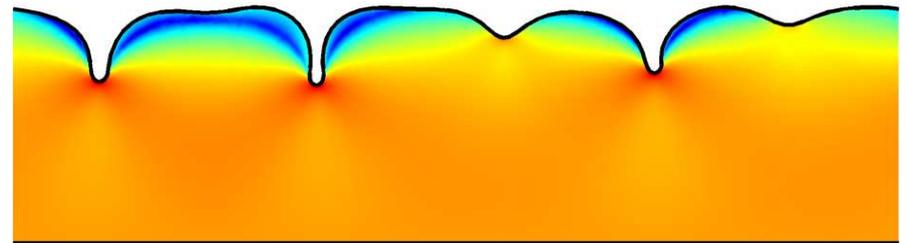
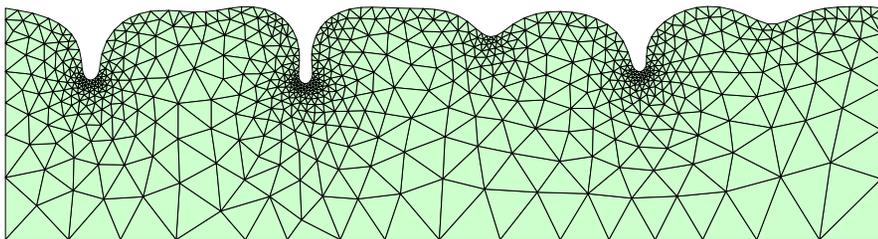


Stress Driven Rearrangement Instabilities

Initial Configuration



Final Configuration, $\sigma = 0.05$



Fluid Dynamics with Moving Boundaries

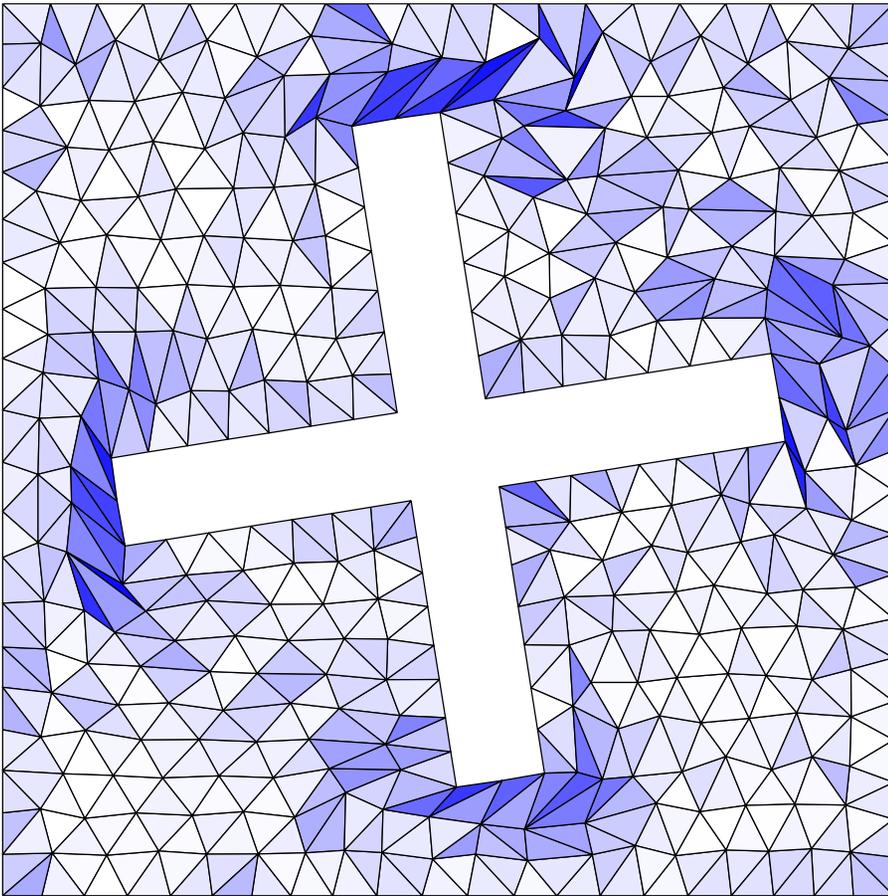
- Solve the incompressible Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \nabla^2 \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \mathbf{f}$$
$$\nabla \cdot \mathbf{u} = 0$$

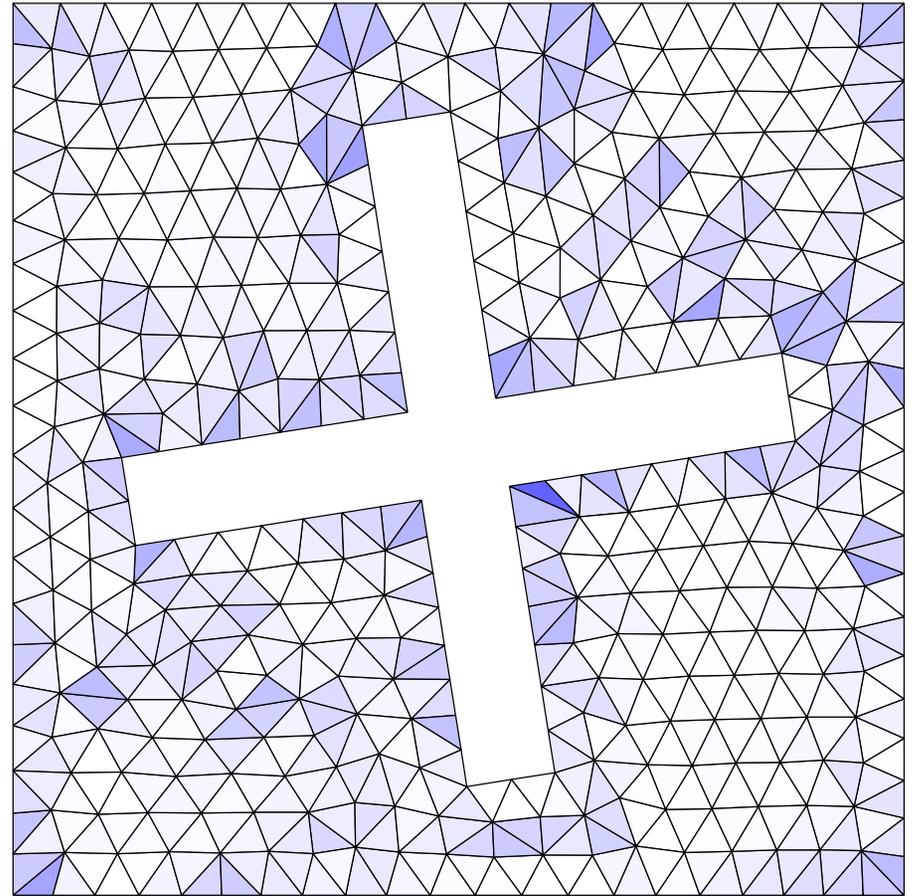
- Spatial discretization using FEM and P2-P1 elements
- Backward Euler for viscous term
- Lagrangian node movement for convective term
 - Improve poor mesh elements using new force equilibrium
 - Interpolate velocity field (better: L_2 -projections, ALE, Space-Time)
- Discrete form of Chorin's projection method for incompressibility

Fluid Dynamics with Moving Boundaries

Lagrangian Node Movement

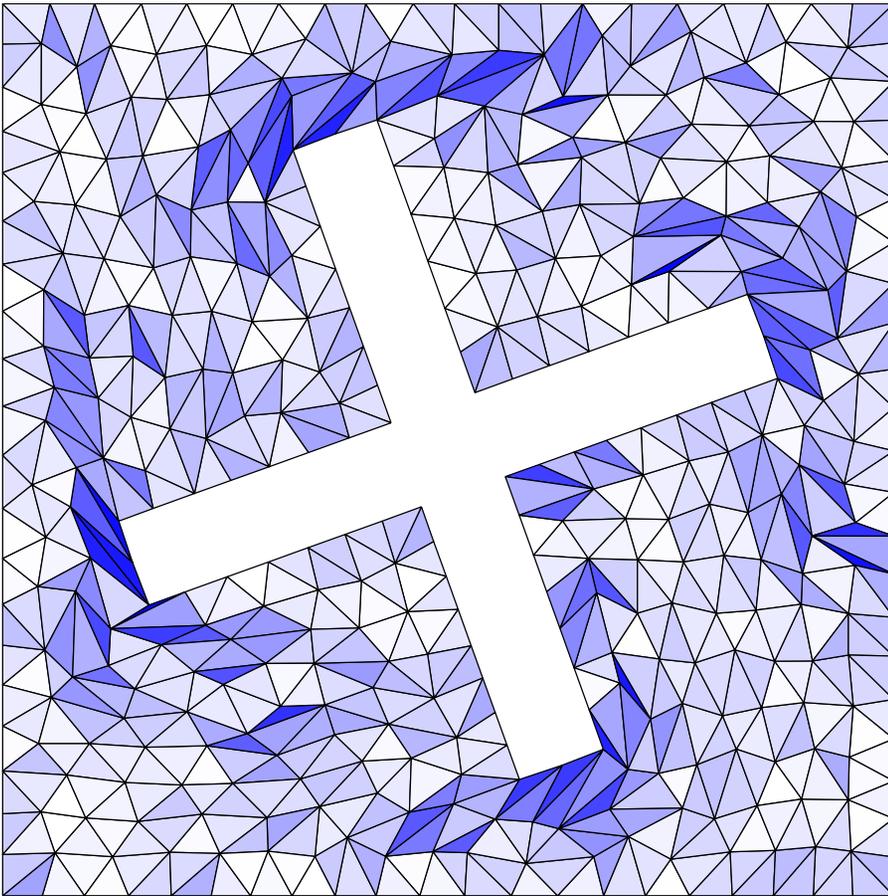


Mesh Improvement

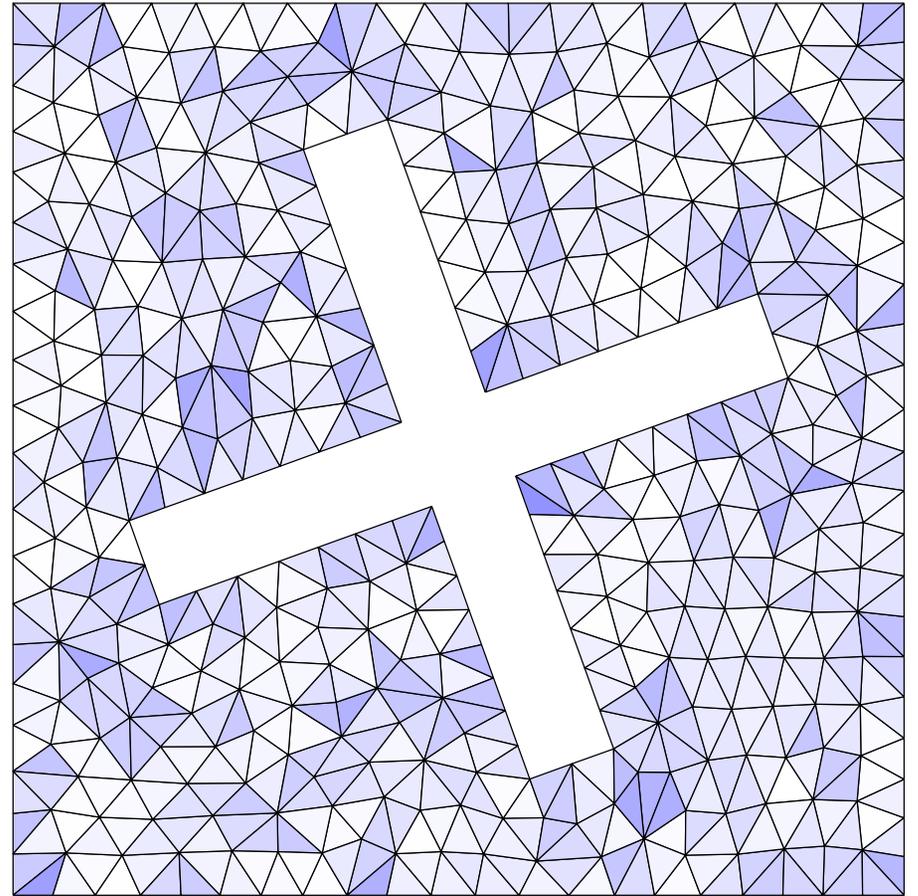


Fluid Dynamics with Moving Boundaries

Lagrangian Node Movement

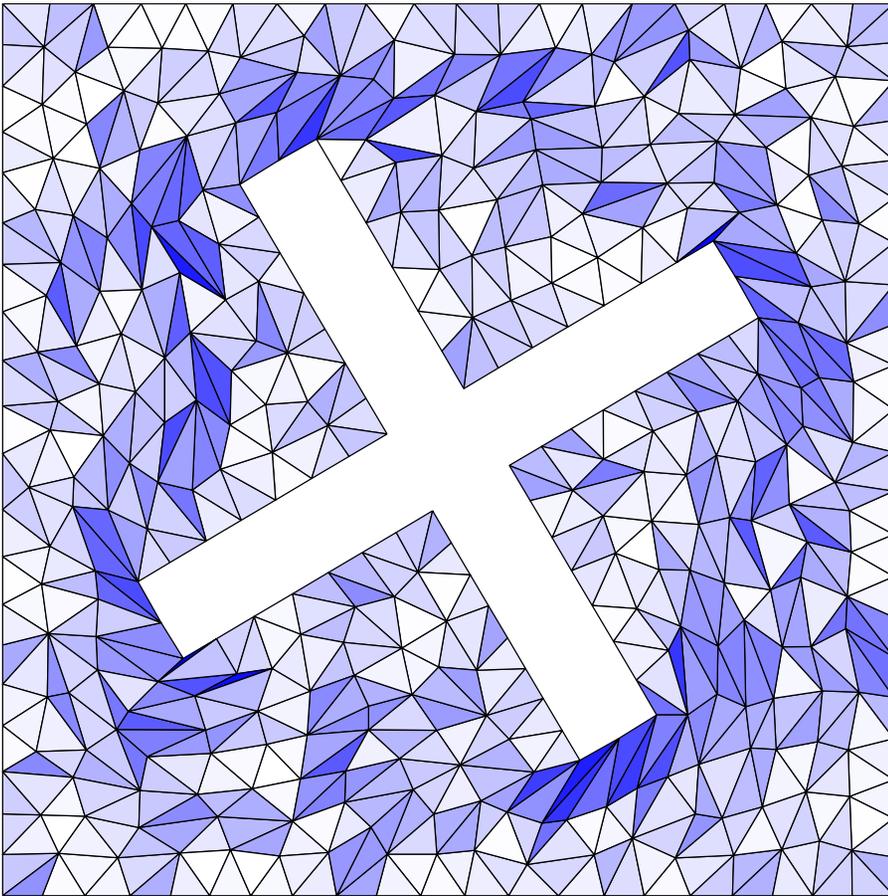


Mesh Improvement

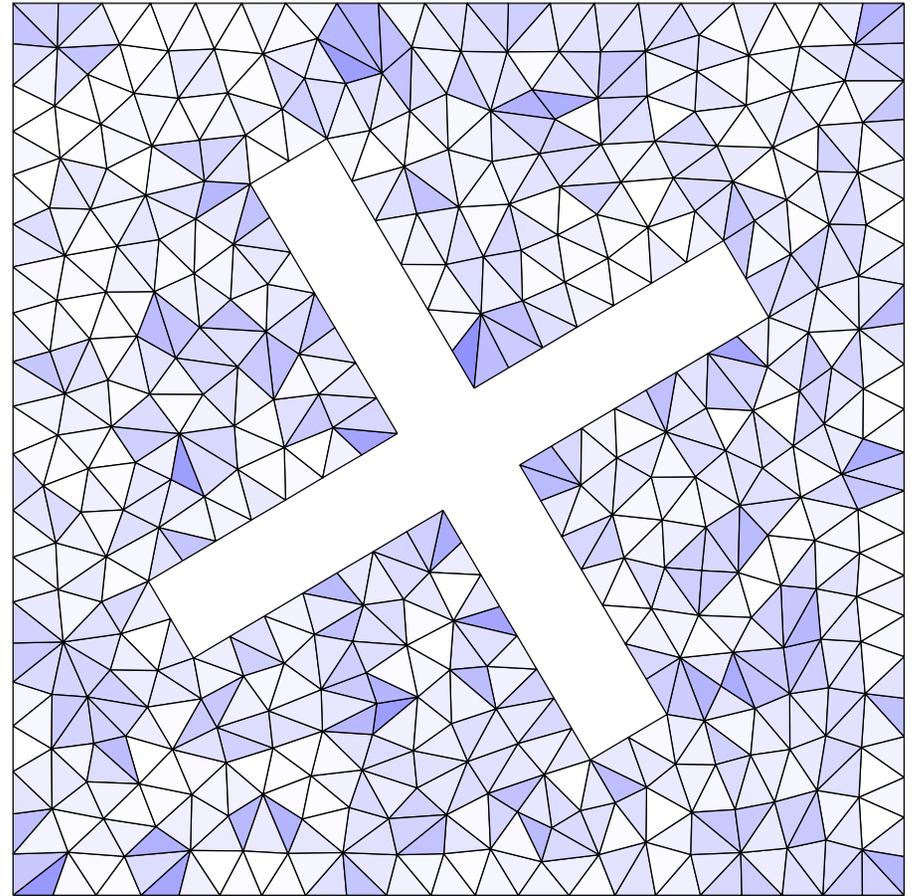


Fluid Dynamics with Moving Boundaries

Lagrangian Node Movement



Mesh Improvement



MATLAB Demo

Meshing Images

- Images are implicit representations of objects
- Segment image to isolate objects from background
 - Active contours (“snakes”), available in standard imaging software
 - Level set based methods [Chan/Vese]
- Apply Gaussian smoothing on binary images (masks)
- Mesh the domain:
 - Find distance function using approximate projections and FMM
 - Find size function from curvature, feature size, and gradient limiting
 - Mesh using our force-based implicit mesh generator
- Extends to three dimensions (MRI/CT scans)

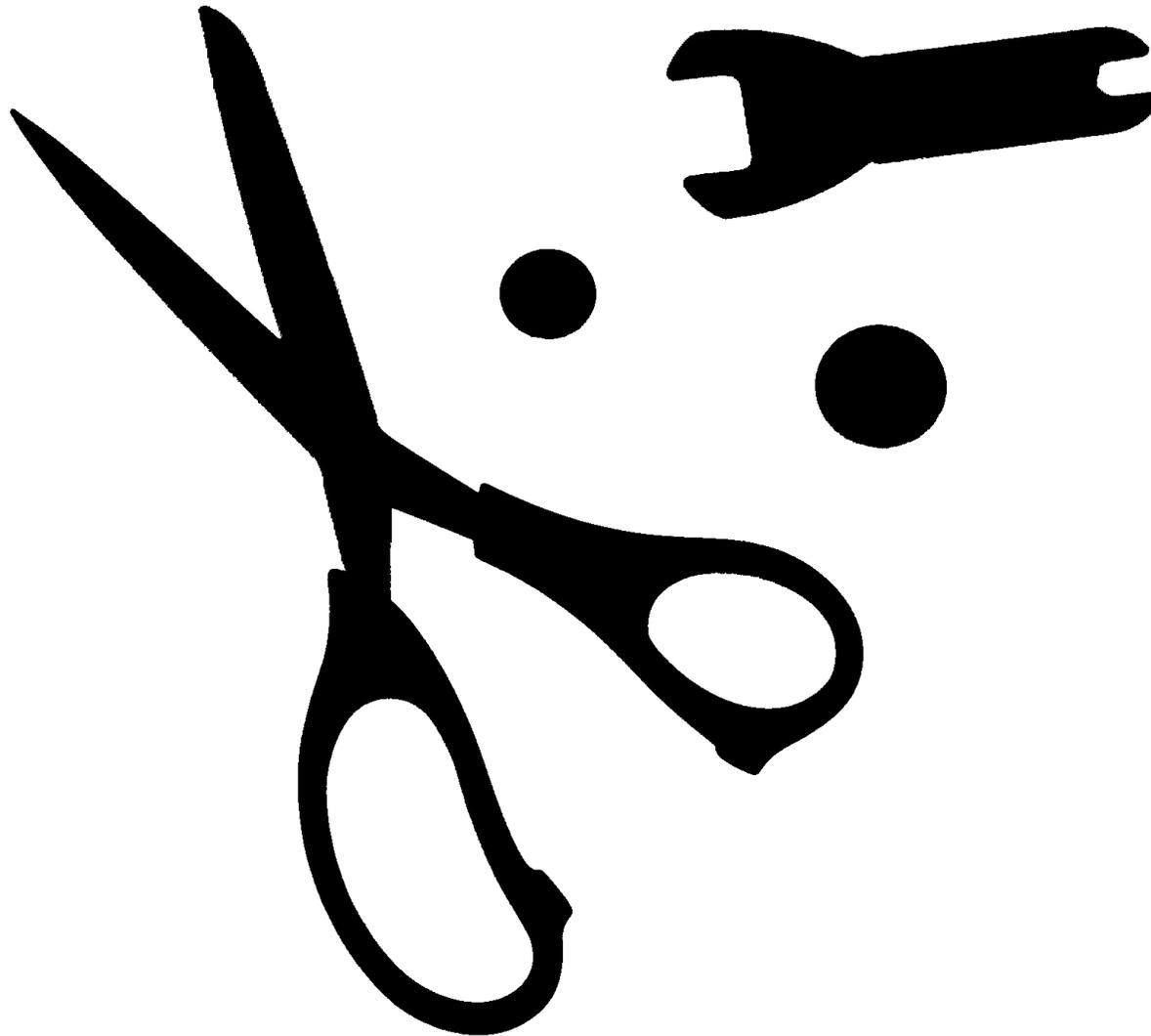
Meshing Images

Original Image



Meshing Images

Segmented Image Mask



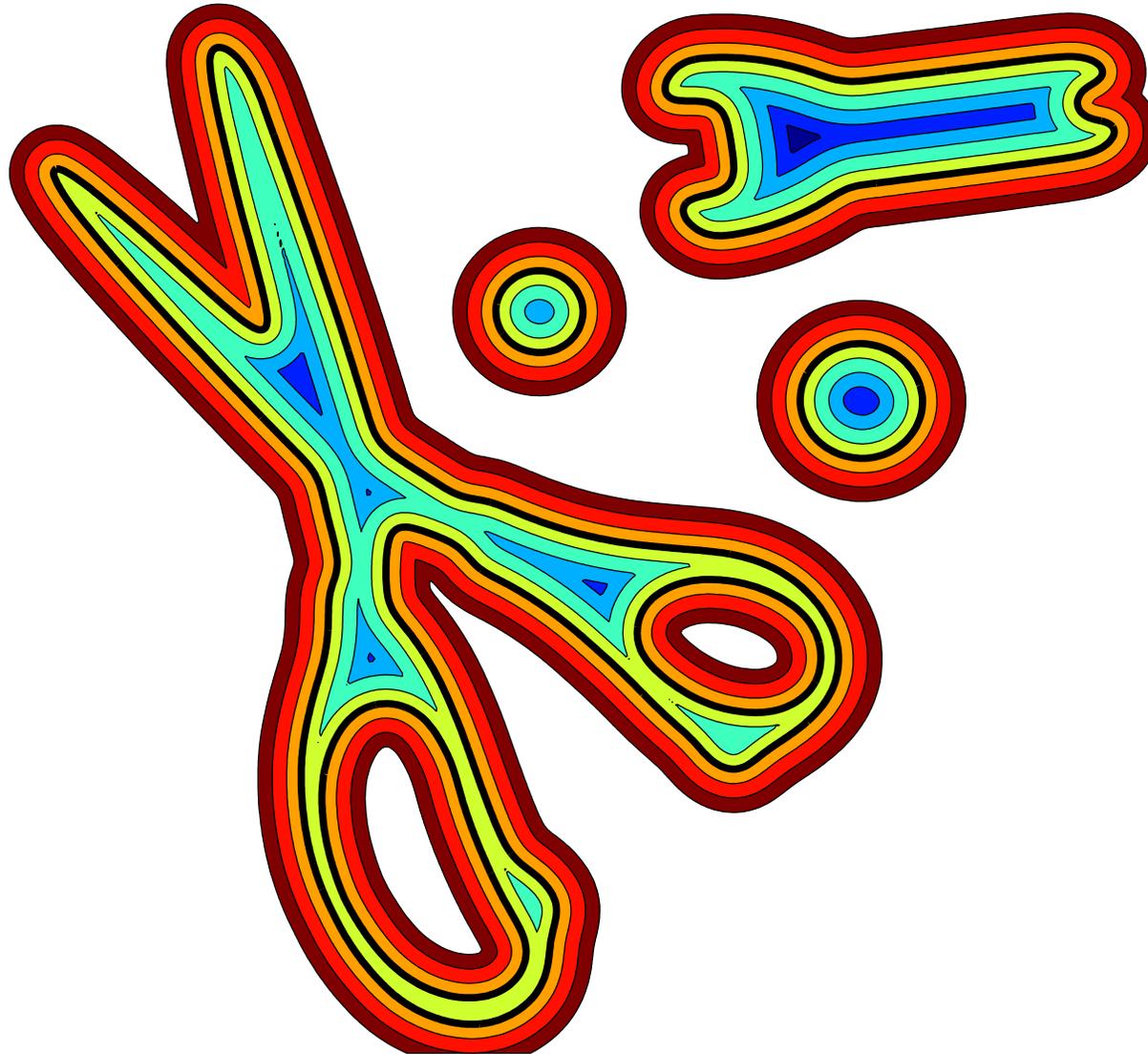
Meshing Images

Mask after Gaussian Smoothing



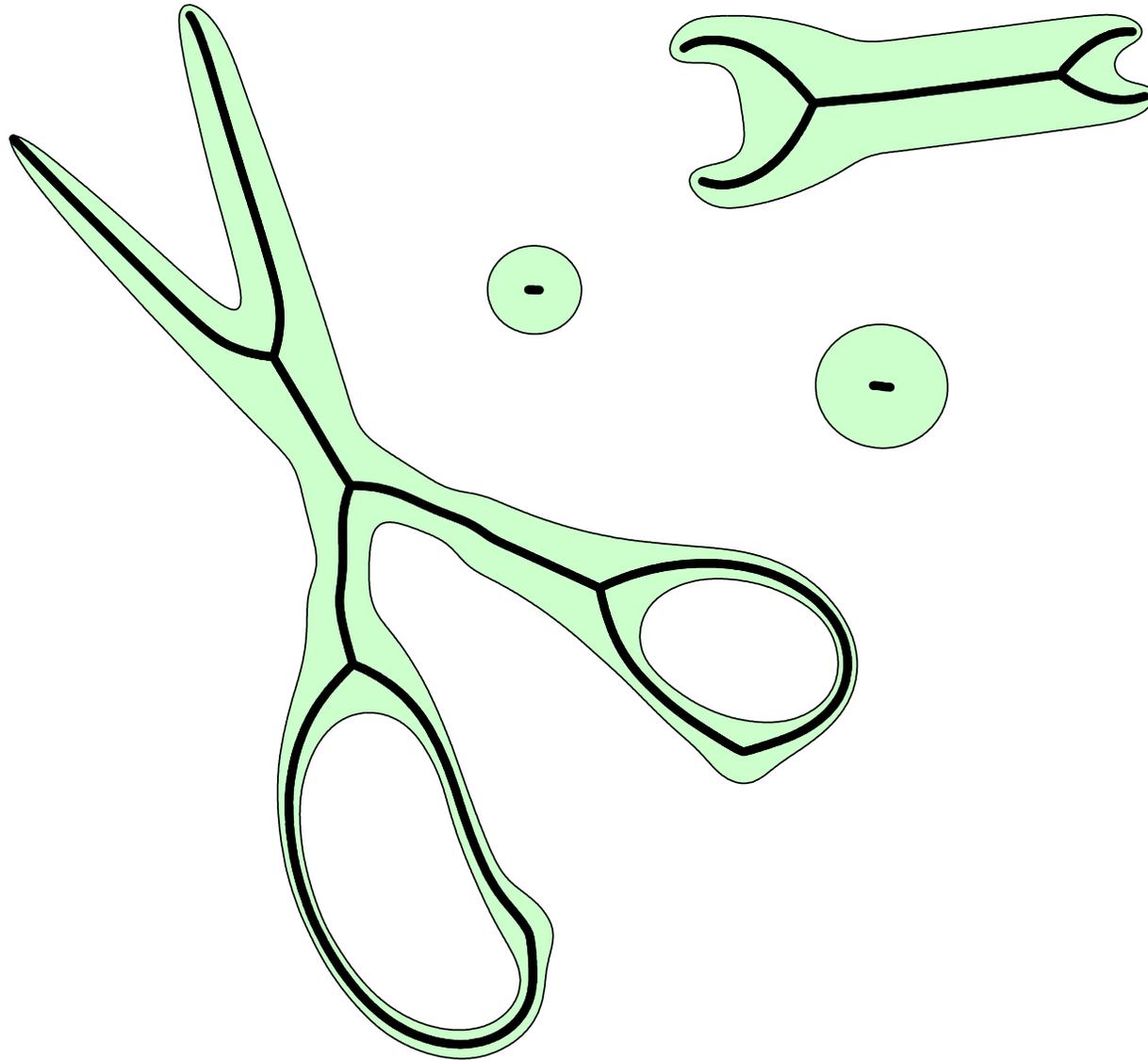
Meshing Images

Distance Function from Approximate Projections and FMM



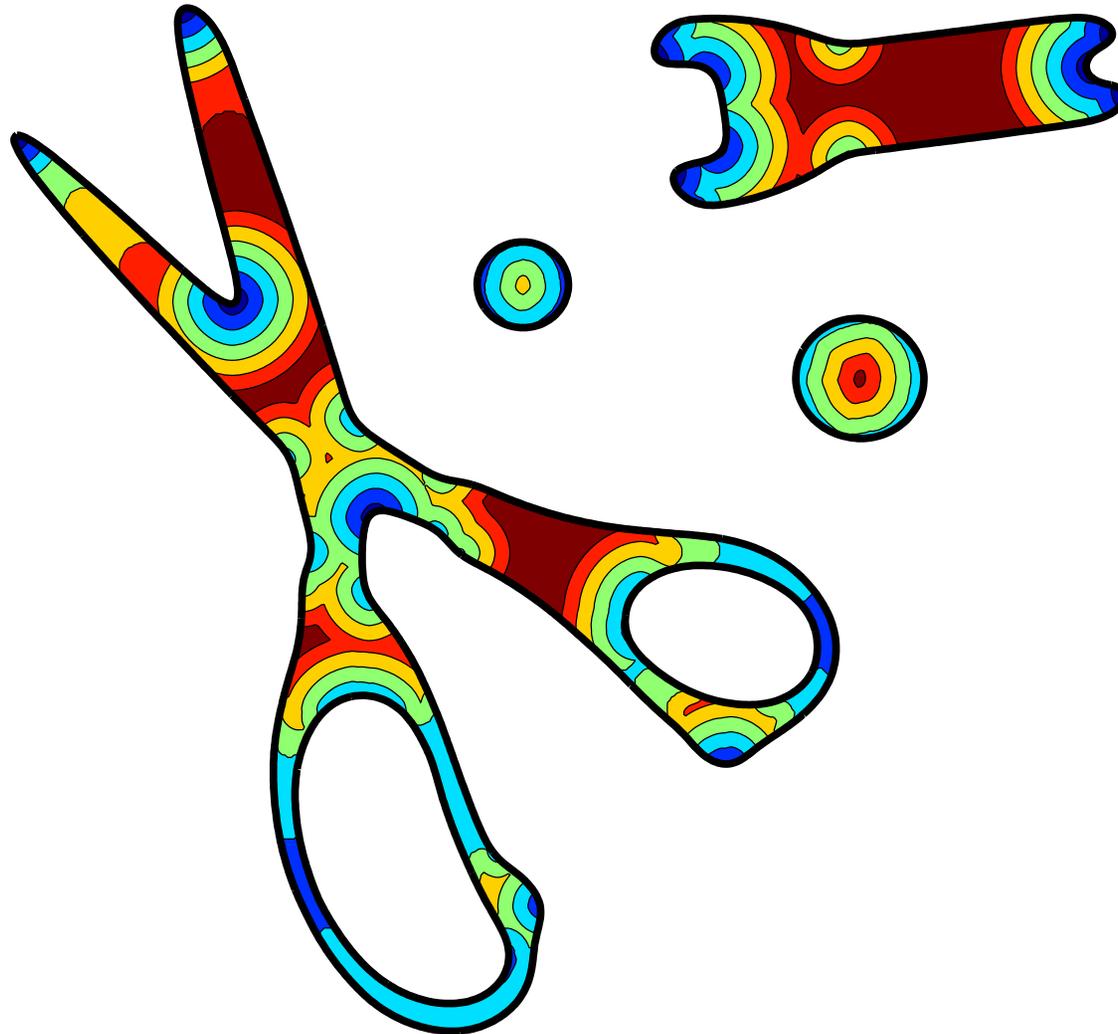
Meshing Images

Medial Axis from Shocks in Distance Functions



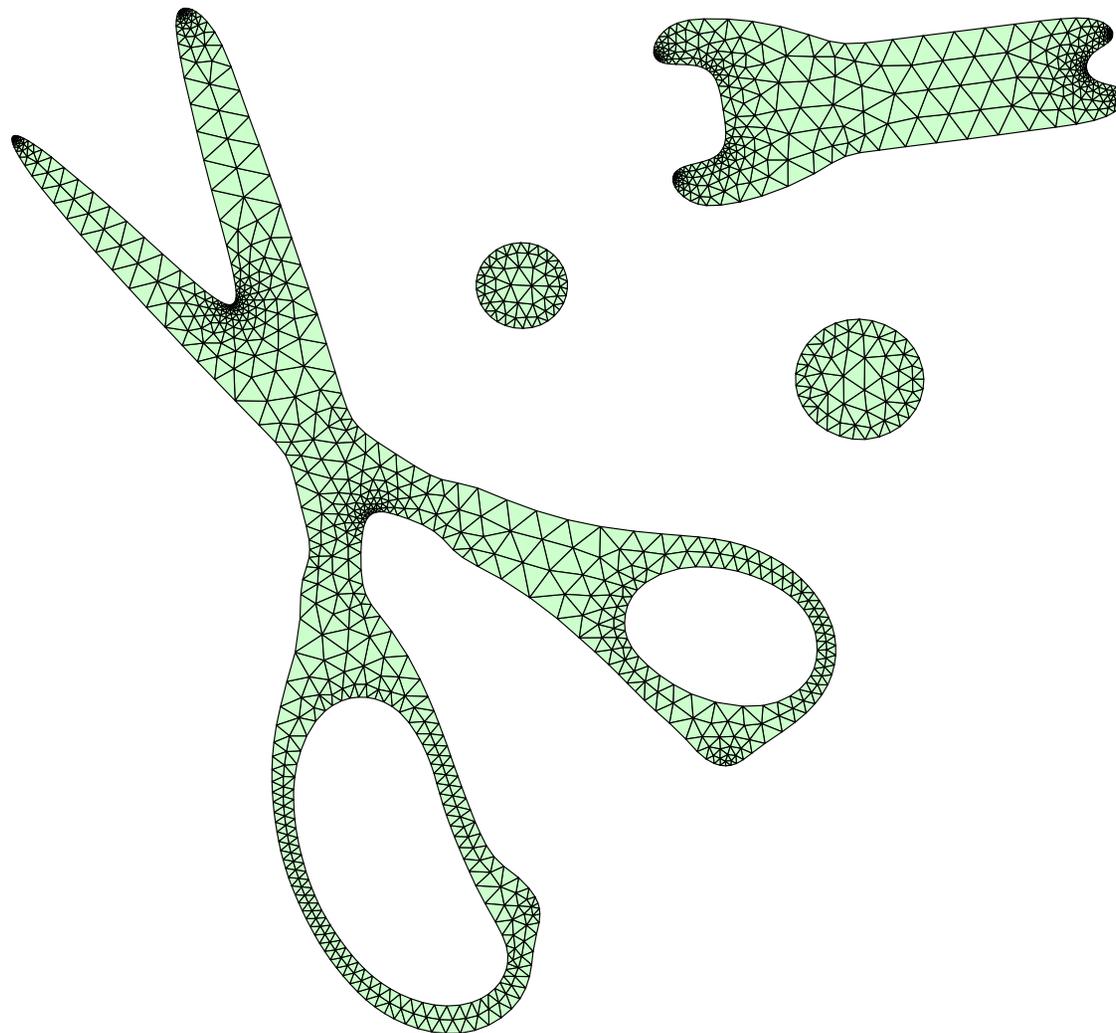
Meshing Images

Curvature, Feature Size, Gradient Limiting \Rightarrow Size Function



Meshing Images

Triangular Mesh



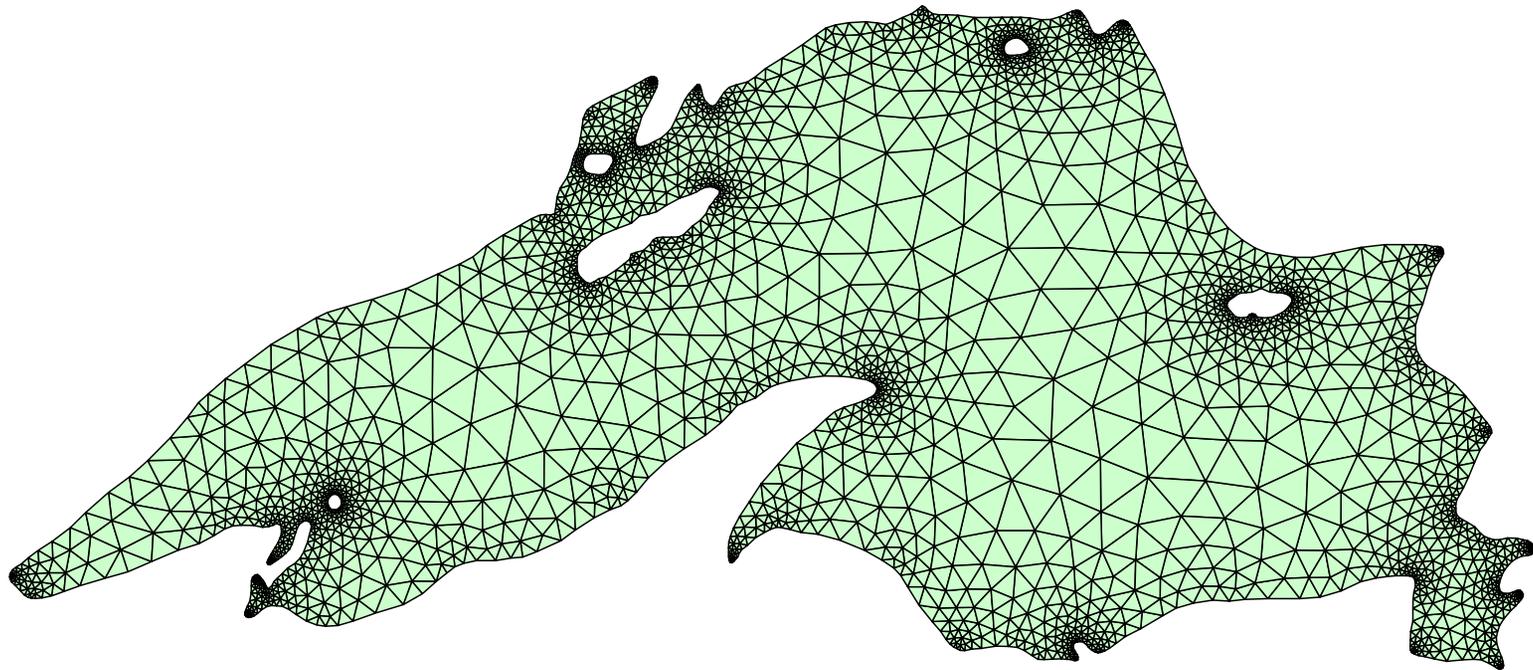
Meshing Images - Lake Superior

Original Image



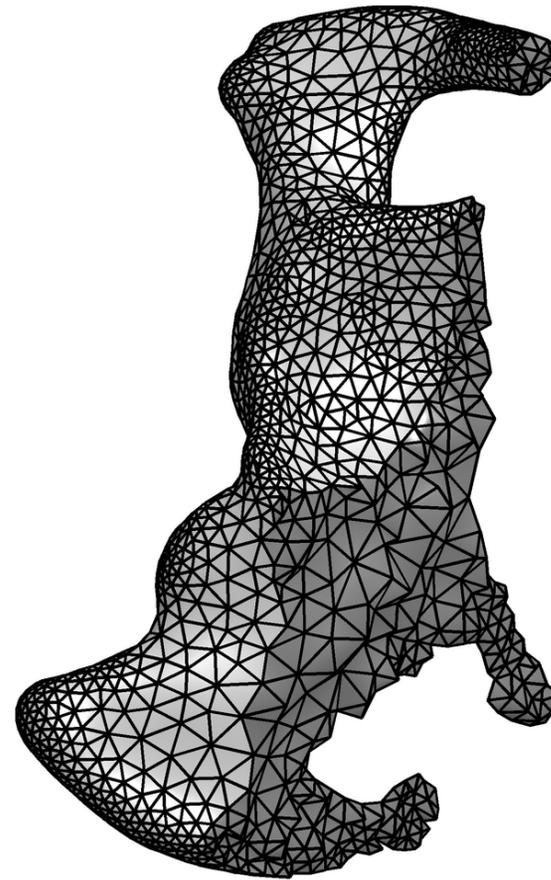
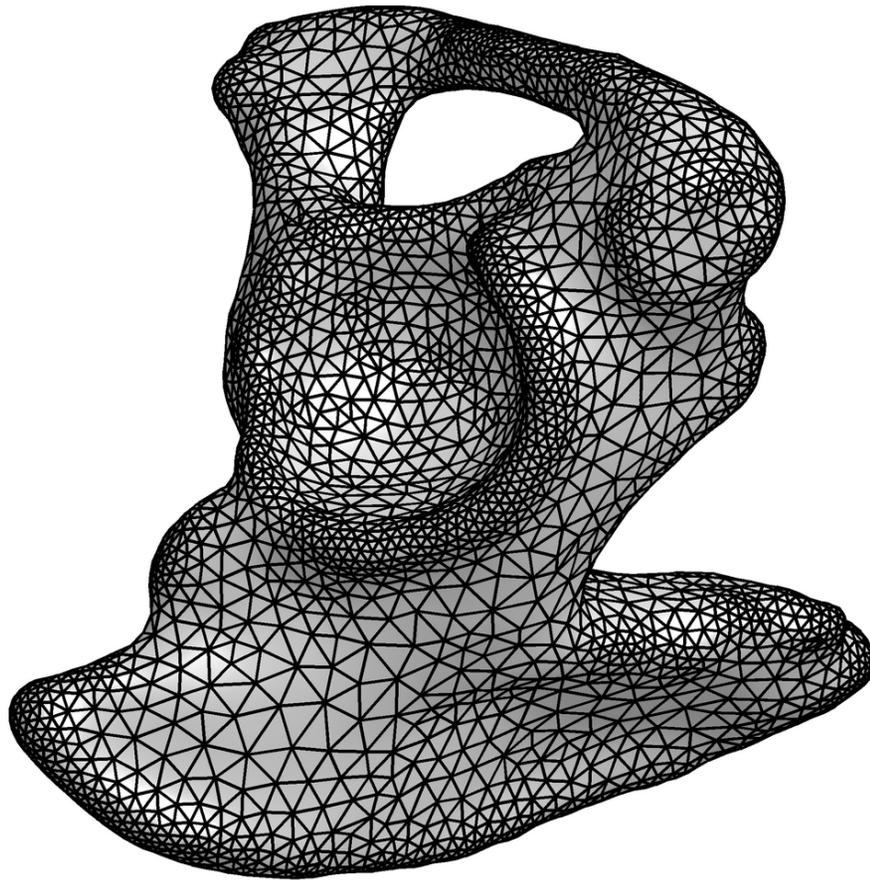
Meshing Images - Lake Superior

Triangular Mesh



Meshing Images - Medical Applications

Volume Mesh of Iliac Bone



Topics

1. Introduction
2. The New Mesh Generator
3. Mesh Size Functions
4. Applications
5. **Conclusions**

Conclusions

- New iterative mesh generator for implicit geometries
- Simple algorithm, high element qualities, works in any dimension
- Automatic computation of size functions, PDE-based gradient limiting
- Combine FEM and level set methods for moving interface problems in shape optimization, structural instabilities, fluid dynamics, etc
- Mesh generation for images (in any dimension)
- Future work: Space-time meshes, sliver removal, performance and robustness improvements, quadrilateral/hexahedral meshes, anisotropic gradient limiting, more applications, ...