Chapter 1 – Vector Spaces

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n-tuples

Example
The set of all n-tuples (a₁, a₂, ..., aₙ) with a₁, a₂, ..., aₙ ∈ F is denoted Fⁿ. This is a vector space with the operations of coordinatewise addition and scalar multiplication: if c ∈ F and u = (a₁, a₂, ..., aₙ) ∈ Fⁿ, v = (b₁, b₂, ..., bₙ) ∈ Fⁿ, then u + v = (a₁ + b₁, a₂ + b₂, ..., aₙ + bₙ), cu = (ca₁, ca₂, ..., caₙ).

u, v are equal if aᵢ = bᵢ for i = 1, 2, ..., n. Vectors in Fⁿ can be written as column vectors

\[
\begin{pmatrix}
a_1 \\ a_2 \\ \vdots \\ a_n
\end{pmatrix}
\]
or row vectors (a₁, a₂, ..., aₙ).

Properties of Vector Spaces

Theorem 1.1 (Cancellation Law for Vector Addition)
If x, y, z are vectors in a vector space V such that x + y = y + z, then x = y.

Corollary 1
The vector 0 described in (VS 3) is unique (the zero vector).

Corollary 2
The vector -x described in (VS 4) is unique (the additive inverse).

Vector Spaces

Definition
A vector space V over a field F is a set with the operations addition and scalar multiplication, so that for each pair x, y ∈ V there is a unique x + y ∈ V, and for each a ∈ F and x ∈ V there is a unique ax ∈ V, such that:

(VS 1) For all x, y ∈ V, x + y = y + x.
(VS 2) For all x, y, z ∈ V, (x + y) + z = x + (y + z).
(VS 3) There exists 0 ∈ V such that x + 0 = x for each x ∈ V.
(VS 4) For each x ∈ V, there exists y ∈ V such that x + y = 0.
(VS 5) For each x ∈ V, 1x = x.
(VS 6) For each a, b ∈ F and each x ∈ V, (ab)x = a(bx).
(VS 7) For each a ∈ F and x, y ∈ V, a(x + y) = ax + ay.
(VS 8) For each a, b ∈ F and each x ∈ V, (a + b)x = ax + bx.

Matrices

Example
An m × n matrix is an array of the form

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]

where aᵢⱼ ∈ F for 1 ≤ i ≤ m, 1 ≤ j ≤ n. The set of all these matrices is denoted Mₘₓₙ(F), which is a vector space with the operations of matrix addition and scalar multiplication: For A, B ∈ Mₘₓₙ(F) and c ∈ F,

\[
(A + B)ᵢⱼ = Aᵢⱼ + Bᵢⱼ \\
(cA)ᵢⱼ = cAᵢⱼ
\]

for 1 ≤ i ≤ m, 1 ≤ j ≤ n.

Functions and Polynomials

Example
Let F(S, F) denote the set of all functions from a nonempty set S to a field F. This is a vector space with the usual operations of addition and scalar multiplication: if f, g ∈ F(S, F) and c ∈ F:

\[(f + g)(s) = f(s) + g(s), \quad (cf)(s) = cf(s) \quad \text{for each} \ s ∈ S.\]

Example
The set P(F) of all polynomials with coefficients in F:

\[f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0\]
is a vector space with the usual operations of addition and scalar multiplication (set higher coefficients to zero if different degrees):

\[f(x) + g(x) = (a_n + b_n)x^n + \cdots + (a_1 + b_1)x + (a_0 + b_0), \quad cf(x) = ca_nx^n + \cdots + ca_1x + ca_0.\]
Subspaces

**Definition**

A subset \( W \) of a vector space \( V \) over a field \( F \) is called a **subspace** of \( V \) if \( W \) is a vector space over \( F \) with the operations of addition and scalar multiplication defined on \( V \).

Note that \( V \) and \( \{0\} \) are subspaces of any vector space \( V \). \( \{0\} \) is called the **zero subspace** of \( V \).

Verification of Subspaces

It is clear that properties (VS 1, 2, 5-8) hold for any subset of vectors in a vector space. Therefore, a subset \( W \) of a vector space \( V \) is a subspace of \( V \) if and only if:

1. \( x + y \in W \) whenever \( x, y \in W \)
2. \( cx \in W \) whenever \( c \in F \) and \( x \in W \)
3. \( W \) has a zero vector
4. Each vector in \( W \) has an additive inverse in \( W \)

Furthermore, the zero vector of \( W \) must be the same as of \( V \), and property 4 follows from property 2 and Theorem 1.2.

A subset \( W \) of a vector space \( V \) is a subspace of \( V \) if and only if:

1. \( 0 \in W \)
2. \( x + y \in W \) whenever \( x, y \in W \)
3. \( cx \in W \) whenever \( c \in F \) and \( x \in W \)

Intersections and Unions of Subspaces

**Theorem 1.4**

Any intersection of subspaces of a vector space \( V \) is a subspace of \( V \).

However, the union of subspaces is not necessarily a subspace, since it need not be closed under addition.

Symmetric and Diagonal Matrices

**Example**

- The transpose \( A^t \) of an \( m \times n \) matrix \( A \) is the \( n \times m \) matrix obtained by interchanging rows and columns of \( A \), that is, \( (A^t)_{ij} = A_{ji} \).
- A symmetric matrix \( A \) has \( A^t = A \) and must be square.
- The set \( W \) of all symmetric matrices in \( M_{n \times n}(F) \) is a subspace of \( M_{n \times n}(F) \).

- An \( m \times m \) matrix \( M \) is a **diagonal matrix** if \( M_{ij} = 0 \) whenever \( i \neq j \).
- The set of diagonal matrices is a subspace of \( M_{n \times n}(F) \).

Linear Combinations

**Definition**

Let \( V \) be a vector space and \( S \) a nonempty subset of \( V \). A vector \( v \in V \) is called a **linear combination** of vectors of \( S \) if there exist a finite number of vectors \( u_1, u_2, \ldots, u_n \) in \( S \) and scalars \( a_1, a_2, \ldots, a_n \) in \( F \) such that

\[
v = a_1u_1 + a_2u_2 + \cdots + a_nu_n.
\]

In this case we also say that \( v \) is a linear combination of \( u_1, u_2, \ldots, u_n \) and call \( a_1, a_2, \ldots, a_n \) the **coefficients** of the linear combination.

Note that \( 0v = 0 \) for each \( v \in V \), so the zero vector is a linear combination of any nonempty subset of \( V \).

Systems of Linear Equations

To solve a system of linear equations, perform the operations:

1. Interchanging the order of any two equations
2. Multiplying any equation by a nonzero constant
3. Adding a constant multiple of one equation to another
to simplify the original system to one with the following properties:
   - The first nonzero coefficient in each equation is one
   - If an unknown is the first unknown with a nonzero coefficient in an equation, then that unknown occurs with a zero coefficient in each other equation
   - The first unknown with a nonzero coefficient in an equation has a larger subscript than the first unknown with a nonzero coefficient in the preceding equations.

It is then easy to solve for some unknowns in terms of the others, or if an equation has the form \( 0 = c \neq 0 \), then the system has no solutions.
Span

**Definition**
Let $S$ be a nonempty subset of a vector space $V$. The span of $S$, denoted $\text{span}(S)$, is the set consisting of all linear combinations of the vectors in $S$. For convenience, we define $\text{span}(\emptyset) = \{0\}$.

**Theorem 1.5**
The span of any subset $S$ of a vector space $V$ is a subspace of $V$. Moreover, any subspace of $V$ that contains $S$ must also contain the span of $S$.

**Definition**
A subset $S$ of a vector space $V$ generates (or spans) $V$ if $\text{span}(S) = V$. In this case, we also say that the vectors of $S$ generate (or span) $V$.

Properties of Bases

**Theorem 1.8**
Let $V$ be a vector space and $\beta = \{u_1, \ldots, u_n\}$ be a subset of $V$. Then $\beta$ is a basis for $V$ if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of $\beta$:

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

for unique scalars $a_1, \ldots, a_n$.

**Theorem 1.9**
If a vector space $V$ is generated by a finite set $S$, then some subset of $S$ is a basis for $V$. Hence $V$ has a finite basis.

Linear Dependence

**Definition**
A subset $S$ of a vector space $V$ is called linearly dependent if there exist a finite number of distinct vectors $u_1, u_2, \ldots, u_n$ in $S$ and scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0.$$  

In this case we also say that the vectors of $S$ are linearly dependent.

**Definition**
A subset $S$ of a vector space that is not linearly dependent is called linearly independent, and the vectors of $S$ are linearly independent.

**Properties of linearly independent sets**
- The empty set is linearly independent
- A set with a single nonzero vector is linearly independent
- A set is linearly independent $\iff$ the only representations of 0 as a linear combination of its vectors are trivial

Basis

**Definition**
A basis $\beta$ for a vector space $V$ is a linearly independent subset of $V$ that generates $V$. The vectors of $\beta$ form a basis for $V$.

**Example**
Since $\text{span}(\emptyset) = \{0\}$ and $\emptyset$ is linearly independent, $\emptyset$ is a basis for the zero vector space.

**Example**
The basis $\{e_1, \ldots, e_n\}$ with $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, \ldots, $e_n = (0, \ldots, 0, 1)$, is called the standard basis for $F^n$.

**Example**
The basis $\{1, x, x^2, \ldots, x^n\}$ is called the standard basis for $P_n(F)$.

The Replacement Theorem

**Theorem 1.10 (Replacement Theorem)**
Let $V$ be a vector space that is generated by a set $G$ containing exactly $n$ vectors, and let $L$ be a linearly independent subset of $V$ containing exactly $m$ vectors. Then $m \leq n$ and there exists a subset $H$ of $G$ containing exactly $n - m$ vectors such that $L \cup H$ generates $V$.

**Corollary 1**
Let $V$ be a vector space having a finite basis. Then every basis for $V$ contains the same number of vectors.
### Dimension

**Definition**

A vector space $V$ is called **finite-dimensional** if it has a basis consisting of a finite number of vectors, this unique number $\dim(V)$ is called the **dimension** of $V$. If $V$ is not finite-dimensional it is called **infinite-dimensional**.

**Corollary 2**

Let $V$ be a vector space with dimension $n$.

(a) Any finite generating set for $V$ contains at least $n$ vectors, and a generating set for $V$ that contains exactly $n$ vectors is a basis for $V$.

(b) Any linearly independent subset of $V$ that contains exactly $n$ vectors is a basis for $V$.

(c) Every linearly independent subset of $V$ can be extended to a basis for $V$.

### Dimension of Subspaces

**Theorem 1.11**

Let $W$ be a subspace of a finite-dimensional vector space $V$. Then $W$ is finite-dimensional and $\dim(W) \leq \dim(V)$. Moreover, if $\dim(W) = \dim(V)$, then $V = W$.

**Corollary**

If $W$ is a subspace of a finite-dimensional vector space $V$, then any basis for $W$ can be extended to a basis for $V$. 