Chapter 3 – Elementary Matrix Operations and Systems of Linear Equations

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Elementary Matrix Operations

Definition
Elementary row/column operations on an \( m \times n \) matrix \( A \):
(1) interchanging any two rows/columns
(2) multiplying any row/column by nonzero scalar
(3) adding any scalar multiple of a row/column to another row/column

Properties of Elementary Operations

Theorem 3.1
Let \( A \in M_{m\times n}(F) \), and \( B \) obtained from an elementary row/column operation on \( A \). Then there exists an \( m \times m / n \times n \) elementary matrix \( E \) s.t. \( B = EA/BE = AE \). This \( E \) is obtained by performing the same operation on \( I_m/I_n \). Conversely, for elementary \( E \), then \( EA/AE \) is obtained by performing the same operation of \( A \) as that which produces \( E \) from \( I_m/I_n \).

Theorem 3.2
Elementary matrices are invertible, and the inverse is an elementary matrix of the same type.

The Rank of a Matrix

Definition
The rank of a matrix \( A \in M_{m\times n}(F) \) is the rank of the linear transformation \( L_A : F^n \to F^m \).

Theorem 3.3
Let \( T : V \to W \) be linear between finite-dimensional \( V, W \) with ordered bases \( \beta, \gamma \). Then \( \text{rank}(T) = \text{rank}([T]_\beta^\gamma) \).

Theorem 3.4
Let \( A \) be \( m \times n \), and \( P, Q \) invertible of sizes \( m \times m, n \times n \). Then
(a) \( \text{rank}(AQ) = \text{rank}(A) \)
(b) \( \text{rank}(PA) = \text{rank}(A) \)
(c) \( \text{rank}(PAQ) = \text{rank}(A) \)

Corollary
Elementary row/column operations are rank-preserving.

Determining the Rank of a Matrix

Theorem 3.5
\( \text{rank}(A) \) is the maximum number of linearly independent columns of \( A \), that is, the dimension of the subspace generated by its columns.

Theorem 3.6
Let \( A \) be \( m \times n \) or rank \( r \). Then \( r \leq m, r \leq n \), and by finite number of elementary row/column operations \( A \) can be transformed into
\[
D = \begin{pmatrix}
I_r & O_1 \\
O_2 & O_3
\end{pmatrix}
\]
where \( O_1, O_2, O_3 \) are zero matrices, that is, \( D_{ij} = 1 \) for \( i \leq r \) and \( D_{ij} = 0 \) otherwise.

Determining the Rank of a Matrix

Corollary 1
Let \( A \) be \( m \times n \) or rank \( r \). Then there exists invertible \( B, C \) of sizes \( m \times m, n \times n \) such that
\[
D = BAC = \begin{pmatrix} I_r & O_1 \\
O_2 & O_3
\end{pmatrix}
\]

Corollary 2
Let \( A \) be \( m \times n \), then
(a) \( \text{rank}(A^t) = \text{rank}(A) \)
(b) \( \text{rank}(A) \) is the maximum number of linearly independent rows, that is, the dimension of the subspace generated by its rows.
(c) The rows and columns of \( A \) generate subspaces of the same dimension, namely \( \text{rank}(A) \)

Corollary 3
Every invertible matrix is a product of elementary matrices.
Rank of Matrix Products

Theorem 3.7
Let $T : V \rightarrow W$ and $U : W \rightarrow Z$ be linear on finite-dimensional $V, W, Z$. Let $A, B$ be matrices such that $AB$ is defined. Then
(a) $\text{rank}(UT) \leq \text{rank}(U)$
(b) $\text{rank}(UT) \leq \text{rank}(T)$
(c) $\text{rank}(AB) \leq \text{rank}(A)$
(d) $\text{rank}(AB) \leq \text{rank}(B)$

Systems of Linear Equations

System of $m$ linear equations in $n$ unknowns:

\[ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
 a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
 a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \]

or

\[ Ax = b \]

with coefficient matrix $A$ and vectors $x, b$:

\[ A = \begin{pmatrix}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 a_{21} & a_{22} & \cdots & a_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}, \quad x = \begin{pmatrix}
 x_1 \\
 x_2 \\
 \vdots \\
 x_n
\end{pmatrix}, \quad b = \begin{pmatrix}
 b_1 \\
 b_2 \\
 \vdots \\
 b_m
\end{pmatrix} \]

Solution Sets

- A solution to the system $Ax=b$: \[ s = \begin{pmatrix}
 s_1 \\
 s_2 \\
 \vdots \\
 s_n
\end{pmatrix} \in \mathbb{F}^n \text{ such that } As = b \]
- The solution set of the system: The set of all solutions
- Consistent system: Nonempty solution set
- Inconsistent system: Empty solution set

Solutions of Systems of Linear Equations

Definition
$Ax = b$ is homogeneous if $b = 0$, otherwise nonhomogeneous.

Theorem 3.8
Let $Ax = 0$ be a homogeneous system of $m$ equations in $n$ unknowns. The set of all solutions to $Ax = 0$ is $K = N(L_A)$, which is a subspace of $\mathbb{F}^n$ of dimension $n - \text{rank}(L_A) = n - \text{rank}(A)$.

Corollary
If $m < n$, the system $Ax = 0$ has a nonzero solution.

Theorem 3.9
Let $K$ be the solution set of $Ax = b$, and let $K_H$ be the solution set of the corresponding homogeneous system $Ax = 0$. Then for any solution $s$ to $Ax = b$:

\[ K = \{s\} + K_H = \{s + k : k \in K_H\} \]

Invertibility and Consistency

Theorem 3.10
If $A$ is invertible then the system $Ax = b$ has exactly one solution $x = A^{-1}b$. Conversely, if the system has exactly one solution then $A$ is invertible.

Theorem 3.11
The system $Ax = b$ is consistent if and only if
\[ \text{rank}(A) = \text{rank}(A|b) \]
Equivalent Systems

**Definition**
Two systems of linear equations are called equivalent if they have the same solution set.

**Theorem 3.13**
For an $m \times n$ linear system $Ax = b$ and an invertible $m \times m$ matrix $C$, the system $(CA)x = Cb$ is equivalent to the original system.

**Corollary**
For a linear system $Ax = b$, if $(A'|b')$ is obtained from $(A|b)$ by a finite number of elementary row operations, then $A'x = b'$ is equivalent to the original system.

Reduced Row Echelon Form

**Definition**
A matrix is in reduced row echelon form if:
(a) Any row containing a nonzero entry precedes any row in which all the entries are zero.
(b) The first nonzero entry in each row is the only nonzero entry in its column.
(c) The first nonzero entry in each row is 1 and it occurs in a column right of the first nonzero entry in the preceding row.

**Example**

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & x \\
1 & 0 & 0 & 0 & x \\
0 & 0 & 1 & 0 & x \\
0 & 0 & 0 & 1 & x
\end{bmatrix}
\]

Gaussian Elimination

**Gaussian Elimination**
Reducing an augmented matrix to reduced row echelon form:
- In the *forward pass*, the matrix is transformed into upper triangular form where first nonzero entry of each row is 1, in a column to the right of the first nonzero entry of preceding rows.
- In the *backward pass* or back-substitution, the matrix is transformed into reduced row echelon form by making the first nonzero entry of each row the only nonzero entry of its column.

**Theorem 3.14**
Gaussian elimination transforms any matrix into its reduced row echelon form.

General Solutions

**Theorem 3.15**
Let $Ax = b$ be a system of $r$ nonzero equations in $n$ unknowns. Suppose $\text{rank}(A) = \text{rank}(A|b)$ and that $(A|b)$ is in reduced row echelon form. Then
(a) $\text{rank}(A) = r$
(b) If the general solution of the form
\[
s = s_0 + t_1 u_1 + t_2 u_2 + \cdots + t_{n-r} u_{n-r}
\]
then $\{u_1, u_2, \ldots, u_{n-r}\}$ is a basis for the solution set of the corresponding homogeneous system, and $s_0$ is a solution to the original system.

Interpretation of the Reduced Row Echelon Form

**Theorem 3.16**
Let $A$ be an $m \times n$ matrix of rank $r > 0$ and $B$ the reduced row echelon form of $A$. Then
(a) The number of nonzero rows in $B$ is $r$
(b) For each $i = 1, \ldots, r$, there is a column $b_j$, of $B$ s.t. $b_{ji} = e_i$
(c) The columns of $A$ numbered $j_1, \ldots, j_r$ are linearly independent.
(d) For each $k = 1, \ldots, n$, if column $k$ of $B$ is $d_1 e_1 + \cdots + d_r e_r$ then column $k$ of $A$ is $d_1 a_{j_1} + \cdots + d_r a_{j_r}$.

**Corollary**
The reduced row echelon form of a matrix is unique.