Chapter 5 – Diagonalization

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Diagonalization

A linear operator $T$ on a finite-dimensional vector space $V$ is **diagonalizable** if and only if there exists an ordered basis $\beta$ for $V$ such that $[T]_{\beta}$ is a diagonal matrix. A square matrix $A$ is diagonalizable if $L_A$ is diagonalizable.

**Definition**

Let $T$ be a linear operator on a vector space $V$. A nonzero vector $v \in V$ is an eigenvector of $T$ if there exists a scalar eigenvalue $\lambda$ corresponding to the eigenvector $v$ such that $T(v) = \lambda v$. Let $A \in M_{n \times n}(F)$. A nonzero vector $v \in F^n$ is an eigenvector of $A$ if $v$ is an eigenvector of $L_A$; that is, if $Av = \lambda v$ for some scalar eigenvalue $\lambda$ of $A$ corresponding to the eigenvector $v$.

**Theorem 5.1**

A linear operator $T$ on a finite-dimensional vector space $V$ is diagonalizable if and only if there exists an ordered basis $\beta$ for $V$ consisting of eigenvectors of $T$. If $T$ is diagonalizable, $\beta = \{v_1, \ldots, v_n\}$ is an ordered basis of eigenvectors of $T$, and $D = [T]_\beta$, then $D$ is a diagonal matrix and $D_{jj}$ is the eigenvalue corresponding to $v_j$ for $1 \leq j \leq n$.

To diagonalize a matrix or a linear operator is to find a basis of eigenvectors and the corresponding eigenvalues.

**Theorem 5.2**

Let $A \in M_{n \times n}(F)$. Then a scalar $\lambda$ is an eigenvalue of $A$ if and only if $\det(A - \lambda I_n) = 0$.

**Definition**

Let $A \in M_{n \times n}(F)$. The polynomial $f(t) = \det(A - tI_n)$ is called the characteristic polynomial of $A$.

**Definition**

Let $T$ be a linear operator on an $n$-dimensional vector space $V$ with ordered basis $\beta$. We define the characteristic polynomial $f(t)$ of $T$ to be the characteristic polynomial of $A = [T]_\beta$: $f(t) = \det(A - tI_n)$.

**Theorem 5.3**

Let $A \in M_{n \times n}(F)$.

(a) The characteristic polynomial of $A$ is a polynomial of degree $n$ with leading coefficient $(-1)^n$.

(b) $A$ has at most $n$ distinct eigenvalues.

**Theorem 5.4**

Let $T$ be a linear operator on a vector space $V$, and let $\lambda$ be an eigenvalue of $T$. A vector $v \in V$ is an eigenvector of $T$ corresponding to $\lambda$ if and only if $v \neq 0$ and $v \in N(T - \lambda I)$.

**Theorem 5.5**

Let $T$ be a linear operator on a vector space $V$, and let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of $T$. If $v_1, \ldots, v_k$ are the corresponding eigenvectors, then $\{v_1, \ldots, v_k\}$ is linearly independent.

**Corollary**

Let $T$ be a linear operator on an $n$-dimensional vector space $V$. If $T$ has $n$ distinct eigenvalues, then $T$ is diagonalizable.

**Definition**

A polynomial $f(t)$ in $P(F)$ splits over $F$ if there are scalars $c, a_1, \ldots, a_n$ in $F$ such that $f(t) = c(t - a_1)(t - a_2) \cdots (t - a_n)$.

**Theorem 5.6**

The characteristic polynomial of any diagonalizable operator splits.
### Multiplicity

**Definition**

Let $\lambda$ be an eigenvalue of a linear operator or matrix with characteristic polynomial $f(t)$. The (algebraic) multiplicity of $\lambda$ is the largest positive integer $k$ for which $(t - \lambda)^k$ is a factor of $f(t)$.

**Diagonalizability**

**Definition**

Let $T$ be a linear operator on a vector space $V$, and let $\lambda$ be an eigenvalue of $T$ having multiplicity $m$. Then $1 \leq \dim(E_\lambda) \leq m$.

**Theorem 5.9**

Let $T$ be a linear operator on a finite-dimensional vector space $V$ such that the characteristic polynomial of $T$ splits. Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of $T$. For $i = 1, \ldots, k$, let $v_i \in E_{\lambda_i}$. If
\[ v_1 + v_2 + \cdots + v_k = 0, \]
then $v_i = 0$ for all $i$.

**Theorem 5.10**

Let $W_1, \ldots, W_k$ be subspaces of finite-dimensional vector space $V$. The following are equivalent:

(a) $V = W_1 \oplus \cdots \oplus W_k$.
(b) $V = \sum_{i=1}^{k} W_i$ and for any $v_1, \ldots, v_k$ s.t. $v_i \in W_i$ (1 \leq i \leq k), if $v_1 + \cdots + v_k = 0$, then $v_i = 0$ for all $i$.
(c) Each $v \in V$ can be uniquely written as $v = v_1 + \cdots + v_k$, where $v_i \in W_i$.
(d) If $\gamma_i$ is an ordered basis for $W_i$ (1 \leq i \leq k), then $\gamma_1 \cup \cdots \cup \gamma_k$ is an ordered basis for $V$.
(e) For each $i = 1, \ldots, k$ there exists an ordered basis $\gamma_i$ for $W_i$ such that $\gamma_1 \cup \cdots \cup \gamma_k$ is an ordered basis for $V$.

**Theorem 5.11**

A linear operator $T$ on finite-dimensional vector space $V$ is diagonalizable \iff $V$ is the direct sum of the eigenspaces of $T$.

### Direct Sums

**Theorem 5.10**

Let $W_1, \ldots, W_k$ be subspaces of finite-dimensional vector space $V$.

### Matrix Limits

**Definition**

Let $L, A_1, A_2, \ldots$ be $n \times p$ matrices with complex entries. The sequence $A_1, A_2, \ldots$ is said to converge to the limit $L$ if
\[ \lim_{m \to \infty} (A_m)_{ij} = L_{ij} \text{ for all } 1 \leq i \leq n \text{ and } 1 \leq j \leq p. \]
If $L$ is the limit of the sequence, we write $\lim_{m \to \infty} A_m = L$.

**Theorem 5.12**

Let $A_1, A_2, \ldots$ be a sequence of $n \times p$ matrices with complex entries that converges to $L$. Then for any $P \in M_{n \times n}(C)$ and $Q \in M_{p \times n}(C)$,
\[ \lim_{m \to \infty} PA_m = PL \text{ and } \lim_{m \to \infty} A_m Q = LQ. \]

**Corollary**

Let $A \in M_{n \times n}(C)$ be such that $\lim_{m \to \infty} A^m = L$. Then for any invertible $Q \in M_{n \times n}(C)$,
\[ \lim_{m \to \infty} (QAQ^{-1})^m = QLQ^{-1}. \]
Existence of Limits

Consider the set consisting of the complex number 1 and the interior of the unit disk: \( S = \{ \lambda \in \mathbb{C} : |\lambda| < 1 \text{ or } \lambda = 1 \} \).

**Theorem 5.13**

Let \( A \) be a square matrix with complex entries. Then \( \lim_{m \to \infty} A^m \) exists \( \iff \) both of the following hold:

(a) Every eigenvalue of \( A \) is contained in \( S \).

(b) If 1 is an eigenvalue of \( A \), then the dimension of the eigenspace corresponding to 1 equals the multiplicity of 1 as an eigenvalue of \( A \).

**Invariance of Subspaces**

**Definition**

Let \( T \) be a linear operator on a vector space \( V \). A subspace \( W \) of \( V \) is called a \( T \)-invariant subspace of \( V \) if \( T(W) \subseteq W \), that is, if \( T(v) \in W \) for all \( v \in W \).

For nonzero \( x \in V \), the subspace \( W = \text{span}(\{x, T(x), T^2(x), \ldots \}) \) is called the \( T \)-cyclic subspace of \( V \) generated by \( x \).

**Theorem 5.21**

Let \( T \) be a linear operator on finite-dimensional \( V \), and let \( W \) be a \( T \)-invariant subspace of \( V \). Then the characteristic polynomial of \( T_W \) divides the characteristic polynomial of \( T \).

**Theorem 5.22**

Let \( T \) be a linear operator on finite-dimensional \( V \), and let \( W \) be the \( T \)-cyclic subspace of \( V \) generated by nonzero \( v \in V \). Let \( k = \dim(W) \). Then

(a) \( \{v, T(v), T^2(v), \ldots, T^{k-1}(v)\} \) is a basis for \( W \).

(b) If \( a_0v + a_1T(v) + \cdots + a_{k-1}T^{k-1}(v) + T^k(v) = 0 \), then the characteristic polynomial of \( T_W \) is \( f(t) = (-1)^k(a_0 + a_1t + \cdots + a_{k-1}t^{k-1} + t^k) \).

**The Cayley-Hamilton Theorem**

**Theorem 5.23** (Cayley-Hamilton)

Let \( T \) be a linear operator on finite-dimensional \( V \), and let \( f(t) \) be the characteristic polynomial of \( T \). Then \( f(T) = T_0 \), the zero transformation. That is, \( T \) “satisfies” its characteristic equation.

**Corollary (Cayley-Hamilton Theorem for Matrices)**

Let \( A \) be an \( n \times n \) matrix, and let \( f(t) \) be the characteristic polynomial of \( A \). Then \( f(A) = O \), the \( n \times n \) zero matrix.