Chapter 6 – Inner Product Spaces

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Inner Products

Definition
An inner product on a vector space V over F is a function that assigns a scalar \( \langle x, y \rangle \) for every \( x, y \in V \), such that for all \( x, y, z \in V \) and \( c \in F \):

(a) \( \langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle \)
(b) \( \langle cx, y \rangle = c \langle x, y \rangle \)
(c) \( \langle x, y \rangle = \overline{\langle y, x \rangle} \) (complex conjugation)
(d) \( \langle x, x \rangle > 0 \) if \( x \neq 0 \)

Example
For \( f, g \in V = C([0,1]) \), an inner product is given by
\[
\langle f, g \rangle = \int_0^1 f(t)g(t) \, dt.
\]

Example
The Frobenius inner product on \( V = \mathbb{F}^{m \times n} \) is defined by
\[
\langle A, B \rangle = \text{tr}(B^*A) \quad \text{for } A, B \in V.
\]

Properties of Inner Product Spaces

Definition
The conjugate transpose or adjoint of \( A \in \mathbb{M}_{m \times n}(F) \) is the \( m \times n \) matrix \( A^* \) such that \( (A^*)_{ij} = \overline{A_{ji}} \) for all \( i, j \).

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Norms

Definition
Let \( V \) be an inner product space. For \( x \in V \), the norm or the length of \( x \) is \( \|x\| = \sqrt{\langle x, x \rangle} \).

Example
For \( x = (a_1, \ldots, a_n) \in V = \mathbb{F}^n \), the Euclidean length is the norm
\[
\|x\| = \left[ \sum_{i=1}^n |a_i|^2 \right]^{1/2}
\]

Theorem 6.2
For an inner product space \( V \) over \( F \) and all \( x, y \in V, c \in F \):

(a) \( \|cx\| = |c| \cdot \|x\| \)
(b) \( \|x\| = 0 \) if and only if \( x = 0 \). In any case, \( \|x\| \geq 0 \).
(c) (Cauchy-Schwarz Inequality) \( |\langle x, y \rangle| \leq \|x\| : \|y\| \)
(d) (Triangle Inequality) \( \|x + y\| \leq \|x\| + \|y\| \)

Orthogonality

Definition
Let \( V \) be an inner product space. Vectors \( x, y \in V \) are orthogonal (perpendicular) if \( \langle x, y \rangle = 0 \). A subset \( S \) of \( V \) is orthogonal if any two distinct vectors in \( S \) are orthogonal. A vector \( x \in V \) is a unit vector if \( \|x\| = 1 \). A subset \( S \) of \( V \) is orthonormal if \( S \) is orthogonal and consists entirely of unit vectors.

Example
Consider the inner product space \( H \) of continuous complex-valued functions defined on \( [0,2\pi] \) with the inner product
\[
\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)} \, dt.
\]
Let \( f_n(t) = e^{int} \) for any integer \( n \), where \( 0 \leq t \leq 2\pi \). Then \( S = \{ f_n : n \text{ is an integer} \} \) is an orthonormal subset of \( H \).
Orthonormal Bases

**Definition**
A subset of an inner product space $V$ is an orthonormal basis for $V$ if it is an ordered basis that is orthonormal.

**Theorem 6.3**
Let $V$ be an inner product space and $S = \{v_1, \ldots, v_k\}$ an orthogonal subset of $V$ consisting of nonzero vectors. If $y \in \text{span}(S)$, then $y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i$.

**Corollary 1**
If in addition $S$ is orthonormal, then $y = \sum_{i=1}^{k} (y, v_i) v_i$.

**Corollary 2**
Let $V$ be an inner product space, and $S$ an orthogonal subset of $V$ consisting of nonzero vectors. Then $S$ is linearly independent.

Representations in Orthonormal Bases

**Theorem 6.5**
Let $V$ be a nonzero finite-dimensional inner product space. Then $V$ has an orthonormal basis $\beta$. Furthermore, if $\beta = \{v_1, \ldots, v_n\}$ and $x \in V$, then

$$x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i.$$

**Corollary**
Let $V$ be a finite-dimensional inner product space with an orthonormal basis $\beta = \{v_1, \ldots, v_n\}$. Let $T$ be a linear operator on $V$, and let $A = [T]_\beta$. Then for any $i$ and $j$, $A_{ij} = \langle T(v_j), v_i \rangle$.

**Definition**
Let $\beta$ be an orthonormal subset (possibly infinite) of an inner product space $V$, and let $x \in V$. The Fourier coefficients of $x$ relative to $\beta$ are the scalars $\langle x, v_i \rangle$, where $y \in \beta$.

Orthonormal Complement

**Theorem 6.6**
Let $W$ be a finite-dimensional subspace of an inner product space $V$, and let $y \in V$. Then there exist unique vectors $u \in W$ and $z \in W^\perp$ such that $y = u + z$. Furthermore, if $\{w_1, \ldots, w_k\}$ is an orthonormal basis for $W$, then $u = \sum_{i=1}^{k} (y, w_i) w_i$.

**Corollary**
The vector $u$ in Thm 6.6 is the unique vector in $W$ that is “closest” to $y$; that is, for any $x \in W$, $\|y - x\| \geq \|y - u\|$, and this inequality is an equality if and only if $x = u$ (the orthogonal projection).

Orthogonal Extension

**Theorem 6.7**
Suppose that $S = \{v_1, \ldots, v_k\}$ is an orthonormal set in an $n$-dimensional inner product space $V$. Then

(a) $S$ can be extended to an orthonormal basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for $V$.

(b) If $W = \text{span}(S)$, then $S_1 = \{v_{k+1}, \ldots, v_n\}$ is an orthonormal basis for $W^\perp$.

(c) If $W$ is any subspace of $V$, then

$$\dim(V) = \dim(W) + \dim(W^\perp).$$

The Adjoint

**Theorem 6.8**
Let $V$ be a finite-dimensional inner product space over $F$, and let $g : V \to F$ be a linear transformation. Then there exists a unique vector $y \in V$ such that $g(x) = \langle x, y \rangle$ for all $x \in V$.

**Theorem 6.9**
Let $V$ be a finite-dimensional inner product space, and let $T$ be a linear operator on $V$. Then there exists a unique function $T^* : V \to V$ such that $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for all $x, y \in V$. Furthermore, $T^*$ is linear.

The linear operator $T^*$ is called the adjoint of $T$. 

Gram-Schmidt Orthogonalization

**Theorem 6.4**
Let $V$ be an inner product space and $S = \{w_1, \ldots, w_n\}$ a linearly independent subset of $V$. Define $S' = \{v_1, \ldots, v_n\}$, where $v_1 = w_1$ and

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j \quad \text{for } 2 \leq k \leq n.$$

Then $S'$ is an orthogonal set of nonzero vectors such that $\text{span}(S') = \text{span}(S)$.

This construction of $\{v_1, \ldots, v_n\}$ is called the Gram-Schmidt process.
### Properties of Adjoints

**Theorem 6.10**
Let $V$ be a finite-dimensional inner product space, and let $\beta$ be an orthonormal basis for $V$. If $T$ is a linear operator on $V$, then 
\[ [T^*]_\beta = [T]_\beta^T. \]

**Corollary**
Let $A$ be an $n \times n$ matrix. Then $A^* = (A^T)^*$.

**Theorem 6.11**
Let $V$ be an inner product space, and $T, U$ linear operators on $V$.

(a) $(T + U)^* = T^* + U^*$
(b) $(cT)^* = cT^*$ for any $c \in F$
(c) $(TU)^* = U^*T^*$
(d) $T^{**} = T$
(e) $I^* = I$

### Normal Operators

**Definition**
Let $T$ be a linear operator on an inner product space $V$. $T$ is normal if $TT^* = T^*T$. An $n \times n$ real or complex matrix $A$ is normal if $AA^* = A^*A$.

**Lemma**
Let $T$ be a linear operator on a finite-dimensional inner product space $V$. If $T$ has an eigenvector, then so does $T^*$. If $\lambda_1, \lambda_2$ are distinct eigenvalues of $T$ then the corresponding eigenvectors $x_1, x_2$ are orthogonal.

**Theorem 6.14 (Schur)**
With $T$ as in the lemma, suppose that the characteristic polynomial of $T$ splits. Then there exists an orthonormal basis $\beta$ for $V$ such that $[T]_\beta$ is upper triangular.

**Definition**
Let $V$ be an inner product space and $T$ a linear operator on $V$. $T$ is self-adjoint (Hermitian) if $T = T^*$. An $n \times n$ real or complex matrix $A$ is self-adjoint (Hermitian) if $A = A^*$.

**Lemma**
Let $T$ be a self-adjoint operator on a finite-dimensional inner product space $V$. Then

(a) Every eigenvalue of $T$ is real.
(b) Suppose that $V$ is a real inner product space. Then the characteristic polynomial of $T$ splits.

**Theorem 6.17**
Let $T$ be a linear operator on a finite-dimensional real inner product space $V$. Then $T$ is self-adjoint if and only if there exists an orthonormal basis $\beta$ for $V$ consisting of eigenvectors of $T$.

### Unitary and Orthogonal Operators

**Definition**
Let $T$ be a linear operator on a finite-dimensional inner product space $V$ over $F$. If $\|T(x)\| = \|x\|$ for all $x \in V$, we call $T$ a unitary operator if $F = \mathbb{C}$ and an orthogonal operator if $F = \mathbb{R}$.

**Theorem 6.18**
Let $T$ be a linear operator on a finite-dimensional inner product space $V$. Then the following statements are equivalent.

(a) $TT^* = T^*T = I$
(b) $(T(x), T(y)) = (x, y)$ for all $x, y \in V$
(c) If $\beta$ is an orthonormal basis for $V$, then $T(\beta)$ is an orthonormal basis for $V$.
(d) There exists an orthonormal basis $\beta$ for $V$ such that $T(\beta)$ is an orthonormal basis for $V$.
(e) $\|T(x)\| = \|x\|$ for all $x \in V$. 

### Properties of Normal Operators

**Theorem 6.15**
With $V$ an inner product space and $T$ a normal operator on $V$:

(a) $\|T(x)\| = \|T^*(x)\|$ for all $x \in V$.
(b) $T - cI$ is normal for every $c \in F$.
(c) If $x$ is an eigenvector of $T$, then $x$ is also an eigenvector of $T^*$. In fact, if $T(x) = \lambda x$, then $T^*(x) = \overline{\lambda} x$.
(d) If $\lambda_1, \lambda_2$ are distinct eigenvalues of $T$ then the corresponding eigenvectors $x_1, x_2$ are orthogonal.
### Unitary and Orthogonal Operators

**Lemma**

Let $U$ be a self-adjoint operator on a finite-dimensional inner product space $V$. If $\langle x, U(x) \rangle = 0$ for all $x \in V$, then $U = T_0$.

**Corollary 1**

Let $T$ be a linear operator on a finite-dimensional real inner product space $V$. Then $V$ has an orthonormal basis of eigenvectors of $T$ with eigenvalues of absolute value 1 if and only if $T$ is both self-adjoint and orthogonal.

**Corollary 2**

Let $T$ be a linear operator on a finite-dimensional complex inner product space $V$. Then $V$ has an orthonormal basis of eigenvectors of $T$ with eigenvalues of absolute value 1 if and only if $T$ is unitary.

### Orthogonal and Unitary Matrices

**Definition**

A square matrix $A$ is called an **orthogonal matrix** if $A^t A = A A^t = I$ and **unitary** if $A^* A = A A^* = I$.

Two matrices $A, B$ are **unitarily [orthogonally] equivalent** if and only if there exists a unitary [orthogonal] matrix $P$ such that $A = P^* B P$.

**Theorem 6.19**

Let $A$ be a complex $n \times n$ matrix. Then $A$ is normal if and only if $A$ is unitarily equivalent to a diagonal matrix.

**Theorem 6.20**

Let $A$ be a real $n \times n$ matrix. Then $A$ is symmetric if and only if $A$ is orthogonally equivalent to a real diagonal matrix.

### Schur and Unitarily/Orthogonally Equivalent Matrices

**Theorem 6.21 (Schur)**

Let $A \in M_{n \times n}(F)$ be a matrix whose characteristic polynomial splits over $F$.

(a) If $F = C$, then $A$ is unitarily equivalent to a complex upper triangular matrix.

(b) If $F = R$, then $A$ is orthogonally equivalent to a real upper triangular matrix.