Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, PDAs, and other electronic devices. You may refer to a single 2-sided sheet of notes. Please write your name on each sheet of paper that you turn in. Don’t trust staples to keep your papers together. Explain your answers as is customary and appropriate. Your paper is your ambassador when it is graded. In this midterm, the scalar field \( F \) will be the field of real numbers unless otherwise specified.

These solutions were written by Ken Ribet.

1. Let \( A \) be the \( 100 \times 100 \) matrix of real numbers whose entry in the \((i, j)\)th place is \( ij \) for \( 1 \leq i, j \leq 100 \). Determine the rank of \( A \). (Figure out what it is and show that the rank is what you say it is.)

Looking over the papers that were handed in, I got the impression that this problem was easy for most students. As people pointed out, the \( j \)th column is \( j \) times the first column, so the span of the set of columns is the span of the first column. This span is 1-dimensional, so the rank is 1. More generally, we can take two vectors \((a_1, \ldots, a_n)\) and \((b_1, \ldots, b_m)\) and form the matrix whose \((i, j)\)th entry is \( a_ib_j \). This is called the outer product of the two vectors. The span of this outer product matrix is 1-dimensional if the vectors are both non-zero and 0-dimensional otherwise.

2. Express \( A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \) as a product of elementary matrices.

The idea is to bring \( A \) to the identity matrix \( I \) by a sequence of elementary row operations. Each operation corresponds to left multiplication by an elementary matrix \( E_i \). When I did this, there were three operations; I got \( E_3E_2E_1A = I \), where \( E_1 \) corresponds to the first operation, and so on. Then \( A = E_1^{-1}E_2^{-1}E_3^{-1} \). The inverses of the \( E_i \) are again elementary matrices, so this is the desired expression. I did this problem on paper before typing the exam and got

\[
A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Let me know if there are misprints or errors here. A final comment: as Tom stressed to me, there is no “best way” to get from \( A \) to \( I \) by elementary operations. Your answer may be different from mine and still be correct. In fact, few students got my answer but most students got answers that are essentially correct.

3. Let \( T : V \to W \) be a linear transformation between finite-dimensional vector spaces and let \( T^\dagger : W^* \to V^* \) be the transpose of \( T \). Establish the equations

\[ \text{nullity} \ T^\dagger + \text{rank} \ T = \dim W, \quad \text{nullity} \ T + \text{rank} \ T^\dagger = \dim V. \]
Show that $T$ is onto if and only if $T^t$ is 1-1, and that $T$ is 1-1 if and only if $T^t$ is onto.

The general formula “nullity $T + \text{rank } T = \dim V$” becomes the second equation once we realize that the rank of $T^t$ and the rank of $T$ are equal. This equality, in matrix-speak, is the statement that the row rank and column rank of a matrix coincide. If we apply the general formula to $T^t$, we get the formula nullity $T^t + \text{rank } T^t = \dim W^*$. Now $W^*$ and $W$ have the same dimension. Also, $T$ and $T^t$ have the same rank (as we’ve seen). Hence we get the first of the two equations. To continue, we observe that $T$ is onto if and only if its rank is the dimension of $W$; by the first equation, this is true if and only if the nullity of $T^t$ is 0, i.e., if and only if $T^t$ is 1-1. Similarly, $T$ is 1-1 if and only if the nullity of $T$ is 0; by the second equation, this is true if and only if rank $T^t = \dim V$. This latter equality holds precisely when $T^t$ is onto.

4. Let $V = \mathbf{P}_2(\mathbb{R})$ be the real vector space of polynomials of degree $\leq 2$. You have proved that the following linear functionals form a basis for the dual space $V^*$: $f_- = \text{evaluation at } -1$, $f_0 = \text{evaluation at } 0$, $f_+ = \text{evaluation at } +1$. Let $T : V \to V^*$ be the linear transformation that maps a polynomial $p(x)$ to the functional $f_p$ defined by the formula

$$f_p(q) = 120 \int_0^1 p(x)q(x) \, dx, \quad q \in V.$$ 

Find the matrix of $T$ with respect to the bases $\{1, x, x^2\}$ of $V$ and $\{f_-, f_0, f_+\}$ of $V^*$.

This problem is related to the most recent quiz, which concerned the fact that $V^*$ has $\{f_-, f_0, f_+\}$ as a basis. This means, in particular, that any linear functional $f$ on $V$ may be written $af_- + bf_0 + cf_+$. We find $a$, $b$ and $c$ by evaluating $f$ on $1$, $x$ and $x^2$, three convenient elements of $V$. Namely, we have $f(1) = a + b + c$, $f(x) = -a + c$ and $f(x^2) = a + c$; note, for instance, that $f_-$ takes the value $-1$ on $x$, that $f_0$ is 0 on $x$, and that $f_+$ is 1 on $x$. We have three simple equations for $a$, $b$ and $c$; solving them, we get

$$a = \frac{f(x^2) - f(x)}{2}, \quad b = f(1) - f(x^2), \quad c = \frac{f(x) + f(x^2)}{2}.$$ 

We now have to tabulate the three values $f(1)$, $f(x)$, $f(x^2)$ in the three cases $f = f_1$, $f = f_x$, $f = f_{x^2}$ that correspond to the basis vectors $p = 1$, $p = x$ and $p = x^2$ of $V$ and then plug in to find $a$, $b$ and $c$:

<table>
<thead>
<tr>
<th></th>
<th>$f(1)$</th>
<th>$f(x)$</th>
<th>$f(x^2)$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$</td>
<td>120</td>
<td>60</td>
<td>40</td>
<td>-10</td>
<td>80</td>
<td>50</td>
</tr>
<tr>
<td>$f_x$</td>
<td>60</td>
<td>40</td>
<td>30</td>
<td>-5</td>
<td>35</td>
<td>35</td>
</tr>
<tr>
<td>$f_{x^2}$</td>
<td>40</td>
<td>30</td>
<td>24</td>
<td>-3</td>
<td>16</td>
<td>27</td>
</tr>
</tbody>
</table>

The matrix that we were supposed to calculate is apparently

$$\begin{pmatrix}
-10 & -5 & -3 \\
80  & 30 & 16 \\
50  & 35 & 27
\end{pmatrix}.$$
1. Label the following statements as TRUE or FALSE, giving a short explanation (e.g., a proof or counterexample). There are six parts to this problem, which continues on page 3.

a. A matrix over a field is invertible if and only if it is a product of elementary matrices.

This is TRUE. We proved the statement in class, so I won’t say more.

b. An $n \times n$ matrix with real entries is diagonalizable when considered as an element of $M_n(\mathbb{C})$ if and only if it is diagonalizable when considered as an element of $M_n(\mathbb{R})$.

This is simply not true. We could consider, for example, the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ over $\mathbb{R}$. Its characteristic polynomial is $t^2 + 1$. Over $\mathbb{R}$, there are no eigenvalues because there are no square roots of $-1$. Over $\mathbb{C}$, the polynomial splits into distinct factors, so the matrix is diagonalizable. It’s similar to the diagonal matrix with $i$ and $-i$ on the diagonal.

c. If a matrix over $F$ has $m$ rows and $n$ columns, the row rank of the matrix is at most $n$.

This is certainly TRUE because the row rank is the column rank; the column rank is at most the number of columns because it’s the dimension of the space spanned by the columns.
d. When $A$ is a square matrix over $F$, the system of linear equations $Ax = b$ has exactly one solution if and only if the corresponding homogeneous system $Ax = 0$ has exactly one solution.

This is again TRUE. The equation $Ax = b$ has exactly one solution precisely when $b$ is in the range of $L_A$ and the nullity of $L_A$ is 0. Because $A$ is a square matrix, $L_A$ has nullity 0 if and only if it is onto and if and only if it is invertible. Hence $Ax = b$ has exactly one solution precisely when the nullity of $L_A$ is 0. (The question of whether or not $b$ is in the range then becomes moot.) Meanwhile, the equation $Ax = 0$ has exactly one solution if and only if the nullity of $L_A$ is 0.

e. An upper-triangular matrix with distinct diagonal entries is diagonalizable.

Let $A$ be the matrix. The statement is TRUE because the characteristic polynomial of $A$ is the product $(t - a_1) \cdots (t - a_n)$, where the $a_n$ are the numbers on the diagonal. Since the characteristic polynomial splits completely and has roots with multiplicity 1, $A$ will be diagonalizable.

f. If $A$ is a $n \times n$ real matrix for which $0 = (A - 2I_n)(A - 3I_n)(A - 4I_n)(A - 5I_n)$, then at least one of the numbers 2, 3, 4, 5 is an eigenvalue of $A$.

TRUE again. If the numbers 2, \ldots, 5 are not eigenvalues of $A$, then each of the factors $A - 2I_n$, \ldots, $A - 5I_n$ is invertible. Accordingly, the product $(A - 2I_n)(A - 3I_n)(A - 4I_n)(A - 5I_n)$ is invertible, which means that it is not 0. (We understand that $n$ is positive because we never consider $0 \times 0$ matrices in the course.)

2. Use mathematical induction to establish a formula for the determinant of an

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & c_1 \\
0 & 0 & \cdots & c_2 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & c_{n-1} & \cdots & 0 & 0 \\
c_n & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

that has zero entries except possibly for the last entry of the first row, the second-to-last entry of the second row, \ldots, the second entry of the second-to-last row, and the first entry of the last row. (Such a matrix might be called “anti-diagonal.”)

Let $D_n$ be the indicated determinant when the size of the matrix is $n$. We have $D_1 = c_1$, $D_2 = -c_1 c_2$, and “so on.” Expanding the determinant along the first column (or last row), we get that $D_n = (-1)^{n+1} c_n D_{n-1}$. Thus $D_n$ is
± the product $c_1 \cdots c_n$, and the question is to see what the sign is. The sign changes when $n$ is even, so you get something like $+$, $-$, $-$, $+$, $+$, $\ldots$. The $n$th sign is different from the previous sign if and only if $n$ is even. As far as I’m concerned, it’s enough that you see this; I’m not super-worried about your writing $D_n = (-1)^{s_n}c_1 \cdots c_n$ with an explicit formula for $s_n$. To get an explicit formula, you can observe that $s_n$ is $2 + \cdots + (n+1)$, or $\frac{n(n+3)}{2}$.

3. Suppose that $A \in M_{n \times n}(F)$ is an invertible matrix over a field $F$. Show that there are $c_0, \ldots, c_{n-1}$ in $F$ such that

$$A^{-1} = c_{n-1}A^{n-1} + c_{n-2}A^{n-2} + \cdots + c_1A + c_0I_n.$$  

(Example: if $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, then $A^{-1} = A + I_3$.)

By the Cayley–Hamilton theorem, $f(A) = 0$, where $f(t)$ is the characteristic polynomial of $A$. The polynomial $(-1)^n f(t)$ starts out $t^n + \cdots$; it looks like $t^n + a_{n-1}t^{n-1} + \cdots + a_0$. Note that $a_0$ is $\pm \det A$, since $f(0) = \det(A - 0 \cdot I_n)$.

By the hypothesis that $A$ is invertible, $a_0$ is non-zero. On dividing by $a_0$ and changing some signs, we see that $A$ satisfies a polynomial of the form $1 - c_0 t - c_1 t^2 - \cdots - c_{n-1} t^n$. This means that $I_n = c_0 A + c_1 A^2 + \cdots + c_{n-1} A^{n-1}$. Multiply by $A^{-1}$ to get the desired expression.

Exhibit a $5 \times 5$ complex matrix whose characteristic polynomial is $t + 2t^2 + 3t^3 + 4t^4 - t^5$.

The answer is given by the displayed matrix on page 316 of the book. The matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

sounds to have the right characteristic polynomial. According to the book, the proof is represented by Exercise 19. As you remember, I worked out the proof in class during the last lecture before spring break.

4. Let $V$ be the real vector space $\mathcal{P}_3(\mathbb{R})$ of polynomials of degree $\leq 3$ with real coefficients. Let $\beta = \{1, x, x^2, x^3\}$ be the standard ordered basis of $V$. Let $T : V \to V^*$ be the linear transformation that sends $p \in V$ to $f_p$, where $f_p(q) = \int_0^1 p(x)q(x) \, dx$. Find $[T]_\beta^{\beta^*}$, where $\beta^*$ is the dual basis of $\beta$.  

110 first midterm—page 3
To find $[T]_{\beta}^{\beta^*}$, we have to write $T(x^i)$ for $i = 0, \ldots, 3$ in terms of the basis $\beta^*$. It is natural to write $\beta = \{v_0, v_1, v_2, v_3\}$ and $\beta^* = \{f_0, f_1, f_2, f_3\}$, where $v_i = x^i$ ($i = 0, \ldots, 3$) and $f_j(v_i) = \delta_{ij}$. If $f$ is an element of $V^*$, we have $f = \sum c_j f_j$ with $c_j = f(v_j)$ for each $j$. For each $i$, $T(v_i)$ is the linear form $q \mapsto \int_0^1 x^i q(x) \, dx$; the value of this form on $v_j$ is $\int_0^1 x^i x^j \, dx = \frac{1}{i+j+1}$. Hence the matrix $[T]_{\beta}^{\beta^*}$ is the matrix $(a_{ij})$, $0 \leq i, j \leq 3$ with $a_{ij} = \frac{1}{i+j+1}$. This matrix is explicitly

\[
\begin{pmatrix}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\
\frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\
\frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7}
\end{pmatrix},
\]

I hope.
Please put away all books, calculators, and other portable electronic devices—anything with an ON/OFF switch. You may refer to a single 2-sided sheet of notes. When you answer questions, write your arguments in complete sentences that explain what you are doing: your paper becomes your only representative after the exam is over.

All vector spaces are finite-dimensional over the field of real numbers or the field of complex numbers.

1. Suppose that $T$ is an invertible linear operator on $V$ and that $U$ is a subspace of $V$ that is invariant under $T$. If $v$ is a vector in $V$ such that $Tv \in U$, show that $v$ is an element of $U$.

Quick solution: Let $u = Tv$. Because the restriction of $T$ to $U$ is invertible, there is a unique $v' \in U$ such that $Tv' = u$. Since $Tv' = Tv$ and $T$ is invertible, we have $v' = v$. Hence we have $v \in U$.

2. Suppose that $T$ is a linear operator on $V$ and that $V$ is an inner-product space. Let $T^*$ be the adjoint of $T$. Show that 0 is an eigenvalue of $T$ if and only if 0 is an eigenvalue of $T^*$.

Quick solution: This problem is the special case of problem 28 where we take $\lambda = 0$. In fact, if we can do this special case, then we get the full statement of problem 28 by replacing $T$ by $T - \lambda I$. To say that 0 is an eigenvalue of an operator is to say that the operator is not invertible. Equivalently, this means that its range is smaller than $V$ and also that its null space is non-zero. Thus $T^*$ has 0 as an eigenvalue if and only if its null space is non-zero, and $T$ has 0 as an eigenvalue if and only if its range is smaller than $V$. To see that these statements are equivalent, we can invoke part (a) of Proposition 6.46 on page 120. Specifically, let $U = T^*$. Then $U$ is $\{0\}$ and only if $U^\perp = V$. These statements follows from the equations $\{0\}^\perp = V$ (everything is perpendicular to 0), $V^\perp = \{0\}$ (only 0 is perpendicular to everything) and $(U^\perp)^\perp = U$ (6.33 on page 112).

3. Let $T$ be a linear operator on $V$. Suppose that there is a non-zero vector $v \in V$ such that $T^3 v = Tv$. Show that at least one of the numbers 0, 1, $-1$ is an eigenvalue of $T$.

Quick solution: Because $v$ is in the null space of $T^3 - T$, this operator is not invertible. However, it is the product $T(T - I)(T + I)$; note that the polynomial $x^3 - x$ factors as $x(x-1)(x+1)$! Because the product is non-invertible, at least one factor is non-invertible. To say that $T$ is non-invertible is to say that 0 is an eigenvalue of $T$. To say that $T - I$ is non-invertible is to say that 1 is an eigenvalue of $T$. To say that $T + I$ is non-invertible is to say that $-1$ is an eigenvalue of $T$. ‘null said.

4. Let $U$ be a subspace of the inner-product space $V$, and let $P = P_U$ be the orthogonal projection of $V$ onto $U$. [For $v \in V$, write $v = u + y$ with $u \in U$ and $y \in U^\perp$. Then $Pv = u$.] Show that $P = P^*$.

Quick solution: We need to establish the equality $\langle Pv, w \rangle = \langle v, Pw \rangle$ for $v, w \in V$. Let $v$ and $w$ be in $V$, and write $v = u + x$ as in the statement of the problem. Similarly, write $w = u' + x'$. Then we need to prove $\langle u, u' + x' \rangle = \langle u + x, x' \rangle$. However, the term on the left is $\langle u, u' \rangle + \langle u, x' \rangle = \langle u, u \rangle$ because $u$ and $x'$ are perpendicular. Similarly, the term on the right simplifies to $\langle u, u' \rangle$.
Please put away all books, calculators, and other portable electronic devices—anything with an ON/OFF switch. You may refer to a single 2-sided sheet of notes. For numerical questions, show your work but do not worry about simplifying answers. For proofs, write your arguments in complete sentences that explain what you are doing. Remember that your paper becomes your only representative after the exam is over.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Possible points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<tr>
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<td>3</td>
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<td>4</td>
<td>6 points</td>
</tr>
<tr>
<td>Total:</td>
<td>30 points</td>
</tr>
</tbody>
</table>

1. a. Use row operations to find the inverse of the matrix

\[
\begin{pmatrix}
-2 & 1 & 0 \\
4 & -3 & 1 \\
1 & 1 & -1
\end{pmatrix}
\]

I’m sure that all of you know how to do this and that most of you will do it correctly. The answer seems to be

\[
\begin{pmatrix}
2 & 1 & 1 \\
5 & 2 & 2 \\
7 & 3 & 2
\end{pmatrix}
\]

b. Let \( A \) be an \( m \times n \) matrix of rank \( m \) and let \( B \) be an \( n \times p \) matrix with rank \( n \). Determine the rank of \( AB \). Prove that your answer is correct.

Think of \( L_A : F^n \rightarrow F^m \) and \( L_B : F^p \rightarrow F^n \). Their ranks are equal to the dimensions of the spaces to which they are mapping. Thus these maps are onto. It follows that the same statement is true for \( L_A \circ L_B = L_{AB} \). In other words, \( AB \) has rank \( m \).

2. Label each of the following statements as TRUE or FALSE. Along with your answer, provide a counterexample, an informal proof or an explanation.

a. The determinant is a linear function \( M_{n \times n}(F) \rightarrow F \).

The best answer is “FALSE.” Indeed, the determinant is a linear function of each of its rows (or columns), but it is not a linear function in general. For example, if you multiply \( A \) by a scalar \( c \), \( \det(A) \) gets multiplied by \( c^n \).
b. Every \( n \times n \) matrix may be written as a product of elementary matrices.

This is true for invertible matrices but is FALSE in general.

c. Let \( V \) be a vector space over the field of complex numbers. If \( \{ f_1, f_2, f_3 \} \) is the basis of \( V^* \) dual to \( \{ v_1, v_2, v_3 \} \), then \( \{ f_1, 2f_2, 3f_3 \} \) is the basis dual to \( \{ v_1, 2v_2, 3v_3 \} \).

As discussed in the review lecture on March 16, when you change a basis of \( V \) by a matrix \( Q \), the dual basis changes by the transpose of the inverse of \( Q \). Here, \( Q \) is the diagonal matrix with diagonal entries 1, 2 and 3; the dual basis changes by the diagonal matrix with entries 1, 1/2 and 1/3. Accordingly, the basis dual to \( \{ v_1, 2v_2, 3v_3 \} \) is \( \{ f_1, (1/2)f_2, (1/3)f_3 \} \), so the statement is FALSE.

d. If \( Ax = 0 \) has exactly one solution, then \( Ax = b \) has exactly one solution.

How to interpret this? There is no statement that \( A \) is square, so we have to think that \( A \) might be rectangular in some way. The hypothesis states in other terms that \( L_A \) is 1-1, but \( L_A \) might not be onto if \( A \) has more rows than columns. Hence the statement is FALSE. To create a specific counterexample, imagine that \( A \) is \( 2 \times 1 \); then it represents two equations in one unknown! These equations might be inconsistent if the two entries in \( b \) are not equal to each other—in this case, there will be no solution.

e. One may find positive integers \( n \) and \( m \) and matrices \( A \in M_{n \times m}(C) \), \( B \in M_{m \times n}(C) \) such that \( AB = I_n \) and \( BA = 0_m \).

This problem was suggested by the GSIs, who proposed the following quick solution, which is based on the well known fact that \( AB \) and \( BA \) have equal traces: If \( A \) and \( B \) are as in the statement of the problem, then \( AB \) has trace \( n \) and \( BA \) has trace 0. Since \( n \neq 0 \) in \( C \), there is no situation like this, so the statement is FALSE.

Another way to see this: If \( A \) and \( B \) are as in the statement of the problem, consider \( ABA \). This is \( A \cdot 0_m = 0_{n \times m} \) and also \( I_n \cdot A = A \). Hence \( A = 0_{n \times m} \), so that \( AB \) can’t be the identity.

f. If \( T : V \to W \) is a linear transformation between finite-dimensional vector spaces and \( v_1, v_2, \ldots, v_k \) are linearly independent vectors of \( V \), then \( T(v_1), T(v_2), \ldots, T(v_k) \) may be extended to a basis of \( W \).

This is clearly FALSE because \( T(v_1), T(v_2), \ldots, T(v_k) \) might happen to be linearly dependent and those won’t be part of a basis.

Amazing, everything is FALSE today!

3. Let \( V \) be the real vector space \( P_3(R) \) of polynomials of degree \( \leq 3 \) with real coefficients. Let \( \beta = \{ 1, x, x^2, x^3 \} \) be the standard ordered basis of \( V \). Let \( T : V \to V^* \) be the linear transformation that sends \( p \in V \) to \( f_p \), where \( f_p(q) = \int_{-1}^{1} p(x)q(x) \, dx \). Find \( [T]^B_{\beta} \), where \( \beta^* \) is the dual basis of \( \beta \).

See problem 4 on the second midterm from my course in spring, 2005:

[http://math.berkeley.edu/~ribet/110/mt2sols.pdf](http://math.berkeley.edu/~ribet/110/mt2sols.pdf)

Here the integral is from \(-1\) to \(1\) instead of \(0\) to \(1\); that should make the matrix have lots of zeros this time.

4. Let \( V \) be a finite-dimensional vector space over a field and let \( W \) be a subspace of \( V \). Suppose that \( v \) is a vector of \( V \) that does not lie in \( W \). Show that there is an element \( f \) of \( V^* \) such that \( f(v) = 1 \) and \( f(w) = 0 \) for all \( w \in W \).

Take a basis \( \{ w_1, \ldots, w_k \} \) of \( W \) and note that the set \( \{ w_1, \ldots, w_k, v \} \) is still linearly independent because \( v \) is not in the span of the \( w_i \). Complete this list to get an ordered basis of all of \( V \): \( \{ v_1, \ldots, v_n \} \), where \( v_1, \ldots, v_k \) are simply \( w_1, \ldots, w_k \) and \( v_{k+1} = v \). Let \( \{ f_1, \ldots, f_n \} \) be the dual basis of \( V^* \) and let \( f = f_{k+1} \). Then \( f(v) = 1 \) because \( v = v_{k+1} \) and \( f = f_{k+1} \). Also, \( f(v_j) = 0 \) if \( j \neq k+1 \). In particular, \( f(w_i) = 0 \) for \( i = 1, \ldots, k \). Hence \( f \) is 0 on the span of the \( w_i \), which is \( W \).
MATH 110, solutions to the mock midterm.

1. Consider the vector space $P(\mathbb{R})$ and the subsets $V$ consisting of those vectors (polynomials) $f$ for which:
   
   (a) $f$ has degree 3,
   
   (b) $2f(0) = f(1)$,
   
   (c) $f(t) \geq 0$ whenever $t \geq 0$,
   
   (d) $f(t) = f(1-t)$ for all $t$.

   In which of these cases is $V$ a subspace of $P(\mathbb{R})$?

   **Solution.**
   
   (a) This is not a subspace of $P(\mathbb{R})$. Indeed, adding or subtracting two polynomials of exact degree 3 may result in a polynomial of a smaller degree, e.g., if $f(x) = x^3 - 3x$ and $g(x) = x^3$, then $g(x) - g(x) = 3x$, and $\deg(g - f) = 1$. (Another observation that shows that $V$ is not a subspace is that the zero polynomial 0 is not in $V$.)

   (b) This is a subspace of $P(\mathbb{R})$. The zero polynomial satisfies the defining condition $2f(0) = f(1)$. Also, if $2f(0) = f(1)$ and $2g(0) = g(1)$, then
   
   $$2(\alpha f + \beta g)(0) = \alpha f(0) + \beta g(0) = \alpha f(1) + \beta g(1) = (\alpha f + \beta g)(1)$$

   for all $\alpha, \beta \in \mathbb{R}$. So $V$ is a subspace of $P(\mathbb{R})$.

   (c) This is not a subspace, the (additive) inverse of a function that is nonnegative over $\mathbb{R}_+$ is nonpositive over $\mathbb{R}_+$. E.g., if $f(x) = x^2$, then $f \in V$, but $-f \notin V$. Thus $V$ is not a subspace of $P(\mathbb{R})$.

   (d) This is a subspace of $P(\mathbb{R})$. Indeed, the zero polynomial satisfies the condition $f(t) = f(1-t)$. If two functions, $f$ and $g$ satisfy this condition, then so are all their linear combinations $\alpha f + \beta g$, i.e.,

   $$(\alpha f + \beta g)(t) = \alpha f(t) + \beta g(t) = \alpha f(1-t) + \beta g(1-t) = (\alpha f + \beta g)(1-t).$$

   So $V$ is a subspace of $P(\mathbb{R})$.

2. Let $A = \begin{bmatrix} 0 & 2 & 3 \\ -1 & 3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 0 \\ 6 & 4 \\ -4 & 6 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $w = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$.

   Do the products $Aw$, $B^t v^t$, $v^t Aw$ exist? Evaluate those that do. Is the set $\{A, B^t\}$ linearly independent?

   **Solution.** The matrix $A$ is of size $2 \times 3$ whereas $w$ is of size $3 \times 1$. Since the inner dimensions agree, the product $Aw$ exists. The sizes of the pair $B^t$, $v^t$ are exactly the same, so the
product $B^t v^t$ exists as well. Now, the product $Aw$ is of size $2 \times 1$, but the size of $v$ is $3 \times 1$, so
the inner dimensions $1$ and $2$ disagree, and the product $vAw = v(Aw)$ does not exist. The
products $Aw$ and $B^t v^t$ are equal to
$$Aw = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \quad B^t v^t = (vB)^t = \begin{bmatrix} -2 & 26 \end{bmatrix}.$$

The set $\{A, B^t\}$ is linearly independent: Assume $\alpha A + \beta B^t = 0_{2 \times 3}$, i.e.,
$$\alpha \begin{bmatrix} 0 & 2 & 3 \\ -1 & 3 & -2 \end{bmatrix} + \beta \begin{bmatrix} -2 & 6 & 4 \\ 0 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ Since the $(1, 1)$ entry is zero, we conclude that $\beta = 0$, and since the $(2, 1)$ entry is zero, we
get $\alpha = 0$. So, the matrices $A$ and $B^t$ are linearly independent.

3. Let $A = \begin{bmatrix} 0 & 2 \\ 2 & -2 \end{bmatrix}$, $\beta = \{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$. Find the representation $[L_A]_\beta$, the dual basis $\beta^*$, and the matrix $[(L_A)^t]_{\beta^*}$.

Solution. The map $L_A$ acts on $\beta$ as follows :
$$L_A\beta = \{\begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}\}.$$ Since
$$\begin{bmatrix} 2 \\ -4 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix} = - \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
we conclude
$$[L_A]_{\beta} = \begin{bmatrix} -3 & -1 \\ -1 & 1 \end{bmatrix}.$$ The dual basis for $\beta$ is
$$f_1(x, y) = -x/2 + y/2, \quad f_2(x, y) = x/2 + y/2,$$
and by the theorem on duals of linear maps,
$$[(L_A)^t]_{\beta} = \begin{bmatrix} -3 & -1 \\ -1 & 1 \end{bmatrix}.$$

4. Let $A : P_n(\mathbb{R}) \to P_n(\mathbb{R}) : (Af)(t) := f(t + 1)$.

Prove that
$$A = I + \frac{D}{1!} + \frac{D^2}{2!} + \cdots + \frac{D^n}{n!},$$
where $D$ is the differentiation operator on $P_n(\mathbb{R})$.

**Proof.** Since any polynomial is infinitely differentiable, we can apply Taylor’s formula of any order to expand $f(t + 1)$ around the point $t$. By using Taylor’s formula of order $n$, we obtain

$$f(t + 1) = f(t) + \frac{f'(t)}{1!} + \frac{f''(t)}{2!} + \cdots + \frac{f^{(n)}(t)}{n!} + \frac{f^{(n+1)}(\xi)}{(n+1)!},$$

where $\xi \in (t, t + 1)$. But if $f$ is a polynomial of degree $n$ or smaller, then $f^{(n+1)}(\xi) = 0$ for any point $\xi \in \mathbb{R}$. So the last term in (1) disappears and we see that

$$Af = (I + \frac{D}{1!} + \cdots + \frac{D^n}{n!})f \quad \text{for all } f \in P_n(\mathbb{R}). \quad \square$$

5. Let $m < n$ and let $f_1, \ldots, f_m$ be linear functionals on an $n$-dimensional space $V$. Prove that there exists a nonzero vector $x \in V$ such that $f_j x = 0$ for all $j = 1, \ldots, m$. What does this result say about solutions of linear equations?

**Proof.** Without loss of generality we may assume that the set $\{f_j : j = 1, \ldots, m\}$ is linearly independent, since the result for linearly dependent functionals will follow from the result for linearly independent functionals.

So, assuming independence, complete $\{f_j : j = 1, \ldots, m\}$ to a basis $\{f_j : j = 1, \ldots, n\}$ of $V^*$, and consider its dual basis $\{u_j : j = 1, \ldots, n\}$ of $V$. Since $f_j u_i = \delta_{ij}$, we see that a nonzero vector $x := u_n$ satisfies the required condition $f_j x = 0$ for all $j = 1, \ldots, m$. \quad \square

Since the action of a linear functional on $\mathbb{R}^n$ is realized as multiplication from the left by a row vector, this result says the following: Any linear homogeneous system with fewer equations than unknowns has a nontrivial solution.

6. Reduce the matrix

$$\begin{bmatrix}
1 & -1 & 4 & 3 & -2 & -2 \\
0 & 2 & 0 & 1 & 1 & 3 \\
-1 & 3 & -4 & -2 & 3 & 5
\end{bmatrix}$$

to its reduced row echelon form. Show all steps.

**Solution.** To create 0 in position $(3, 1)$, add the first row to the third row. This gives

$$\begin{bmatrix}
1 & -1 & 4 & 3 & -2 & -2 \\
0 & 2 & 0 & 1 & 1 & 3 \\
0 & 2 & 0 & 1 & 1 & 3
\end{bmatrix}.$$ 

To create zero in position $(3, 2)$, subtract the second row from the third to get

$$\begin{bmatrix}
1 & -1 & 4 & 3 & -2 & -2 \\
0 & 2 & 0 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$
To make the $(2, 2)$ entry equal to 1, divide the second row by 2 to get
\[
\begin{bmatrix}
1 & -1 & 4 & 3 & -2 & -2 \\
0 & 1 & 0 & 1/2 & 1/2 & 3/2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Finally, to create 0 in position $(1, 2)$, add the second row to the first row to obtain
\[
\begin{bmatrix}
1 & 0 & 4 & 7/2 & -3/2 & -1/2 \\
0 & 1 & 0 & 1/2 & 1/2 & 3/2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

This matrix is in reduced row echelon form.
1. (6 points each) Determine whether each of the following assertions is true or false. Give a brief explanation for each answer (full proof is not required).

(a) If \( D: P_3(\mathbb{R}) \to \mathbb{R} \) is linear and satisfies \( D(1) = 0, \ D(x) = 1, \ D(x^2) = 2, \ \text{and} \ \ D(x^3) = 3 \), then \( D(f(x)) = f'(1) \) for all \( f(x) \in P_3(\mathbb{R}) \).

\[
\text{True. The linear transformations } D \text{ and } T(5(x)) = 5'(1) \text{ agree on the basis } \{1, x, x^2, x^3\} \text{ of } P_3(\mathbb{R}), \text{ therefore they are equal.}
\]

(b) There exist (non-empty) matrices \( A \) and \( B \) such that \( AB = I \) and \( BA = 0 \).

\[
\text{False. Solution 1: by homework, } \text{tr}(AB) = \text{tr}(BA). \text{ But } \text{tr}(I_n) = n \text{ and } \text{tr}(0) = 0.
\]

\[
\text{Solution 2: } L_A L_B = I \text{ implies } L_A \text{ is onto, and therefore } R(L_A) \neq \{0\}, \text{ and } L_B \text{ is 1-to-1, from which it follows that } L_B(R(L_A)) \neq \{0\}. \text{ But } L_B(R(L_A)) = R(L_B), \text{ so this implies } L_B L_A \neq 0.
\]

(c) If \( \{v_1, \ldots, v_n\} \) is a basis of \( V \), \( \{w_1, \ldots, w_n\} \) is a basis of \( W \), and \( T: V \to W \) is a linear transformation such that \( T(v_i) = w_i \) for all \( i \), then \( T \) is an isomorphism.

\[
\text{True. Let } \beta = \{v_1, \ldots, v_n\}, \ \gamma = \{w_1, \ldots, w_n\}. \text{ The matrix } A T \text{ is then } [T]_\beta^\gamma = I_n \text{ by hypothesis. Since } I_n \text{ is an invertible matrix, } T \text{ is invertible. Alternatively: } T \text{ is onto since the elements } T(v_i) \text{ span } W, \text{ and an onto linear transformation between spaces with } \dim(V) = \dim(W) \text{ is invertible.}
\]

(d) For every vector space \( V \), the set of linear transformations \( T: V \to V \) such that \( T^2 = 0 \) is a subspace of \( L(V, V) \).

\[
\text{False. For a counterexample, take } V = \mathbb{R}^2, A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \text{ Then } L_A^2 = 0, L_B^2 = 0, \text{ but } (L_A + L_B)^2 \neq 0 \text{ since } A + B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = I_2.
\]

(e) There exists a sequence of row and column operations that transforms

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\text{into \quad}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\text{False. The first matrix has rank 3 and the second has rank 2. But row and column operations preserve rank.}
\]
2. (25 points) Suppose $V$ is a finite-dimensional vector space, $T: V \to V$ is a linear transformation, and $\beta$ and $\gamma$ are ordered bases of $V$. Let $P$ be the change of coordinate matrix such that $P[x]_\beta = [x]_\gamma$ for all $x \in V$. Express each of the matrices $[T]_\gamma$, $[T]_\beta^\gamma$ and $[T]_\beta^\gamma$ in terms of $[T]_\beta$, $P$ and $P^{-1}$.

We have $P = [I]_\gamma^\beta$, $P^{-1} = [I]_\beta^\gamma$.

Therefore $[T]_\gamma = [I]_\gamma^\beta [T]_\beta^\gamma [I]_\gamma^\beta = P[T]_\beta P^{-1}$,

$[T]_\beta^\gamma = [I]_\beta^\gamma [T]_\beta^\gamma = P[T]_\beta$,

$[T]_\beta^\gamma = [I]_\beta^\gamma [I]_\beta^\gamma = [T]_\beta^\gamma P^{-1}$.

3. (20 points) Find $A^9$, where

$$A = \begin{pmatrix} \cos(\pi/5) & -\sin(\pi/5) \\ \sin(\pi/5) & \cos(\pi/5) \end{pmatrix}.$$

In the standard basis $\beta$ of $\mathbb{R}^2$, $A$ is the matrix $[R_{\pi/5}]_\beta$ of the rotation through angle $\pi/5$. Then

$$A^9 = [R_{9\pi/5}]_\beta = [R_{9\pi/5}]_\beta = [R_{9\pi/5}]_\beta = \begin{pmatrix} \cos(9\pi/5) & \sin(9\pi/5) \\ -\sin(9\pi/5) & \cos(9\pi/5) \end{pmatrix}.$$
4. (25 points) Invert the matrix

\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
-3 & -1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
3 & 1 & 2 & 0
\end{pmatrix}
\]

You need not show every step, but you should indicate enough so we can see your method.

\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
-3 & -1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
3 & 1 & 2 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

- switch rows

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

- add multiples of row 1 to rows 2 and 4

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

- add multiples of rows 2 and 3 to row 4

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

- multiply row 2 by -1 and add row 4 to it

The inverse matrix is therefore

\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
-2 & 0 & 3 & 1 \\
1 & 0 & 0 & 0 \\
-2 & 1 & 0 & 1
\end{pmatrix}
\]

Since you can, and should, check your answer by multiplying this by the original matrix, we will not be very generous about arithmetic errors when grading this problem.