2.6.13 For every subset $S$ of $V$, define the *annihilator* $S^0$ of $S$ as

$$S^0 = \{ f \in V^* : f(x) = 0 \text{ for all } x \in S \}.$$

(a) *Prove that $S^0$ is a subspace of $V^*$.*

By Theorem 1.3, since:

1. Clearly the zero functional $f = \theta \in S^0$.
2. For $f, g \in S^0$, we have $(f + g)(x) = f(x) + g(x) = 0 + 0 = 0$ for all $x \in S$, so $f + g \in S^0$.
3. For $f \in S^0$ and $c \in F$, we have $(cf)(x) = cf(x) = c0 = 0$ for all $x \in S$, so $cf \in S^0$.

(b) *If $W$ is a subspace of $V$ and $x \notin W$, prove that there exists $f \in W^0$ such that $f(x) \neq 0$.*

Consider a basis for $W$, $\beta_W = \{v_1, \ldots, v_k\}$, and its extension onto $V$, $\beta = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ with dual basis $\beta^* = \{f_1, \ldots, f_k, f_{k+1}, \ldots, f_n\}$. Note that $f_{k+1}, \ldots, f_n \in W^0$, since any $w \in W$ can be written $w = \sum_{i=1}^{k} a_i v_i$ and $f_j(v_i) = 0$ for $1 \leq i \leq k$ and $k + 1 \leq j \leq n$. But since $x \notin W$, it can be written $x = \sum_{i=1}^{n} a_i v_i$ with at least one $a_\ell \neq 0$ for $k + 1 \leq \ell \leq n$. Then $f_\ell(x) = a_\ell \neq 0$. 