Matrix-Vector Multiplication

- Matrix-vector product \( b = Ax \)

\[
b_i = \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, \ldots, m
\]

- Linear mapping \( x \mapsto Ax \)

\[
A(x + y) = Ax + Ay \\
A(\alpha x) = \alpha Ax
\]

- Every linear map can be expressed as a matrix-vector product
Linear Combination of Columns

- Columns \( a_1, a_2, \ldots, a_n \) of \( A \):

\[
A = \begin{bmatrix}
a_1 & a_2 & \cdots & a_n
\end{bmatrix}
\]

- Alternative view of matrix-vector product:

\[
b = Ax = \sum_{j=1}^{n} x_j a_j = x_1 \begin{bmatrix} a_1 \end{bmatrix} + x_2 \begin{bmatrix} a_2 \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_n \end{bmatrix}
\]

- \( b \) is a linear combination of the columns of \( A \)
Matrix-Matrix Multiplication

- Matrix-matrix product $B = AC$

$$b_{ij} = \sum_{k=1}^{m} a_{ik}c_{kj}$$

- Matrix-vector product for each column of $C$

- Each column of $B$ is a linear combination of the columns of $A$
Range, Nullspace, and Rank

● The range or column space of $A$:

$$\text{range}(A) = \text{All linear combinations of the columns of } A$$

$$= \text{The space spanned by the columns of } A$$

$$= \text{All vectors that can be expressed as } Ax$$

● The nullspace of $A$:

$$\text{null}(A) = \text{All solutions to } Ax = 0$$

● Dimension of space = number of vectors in a basis

● The column rank of $A$ is the dimension of the column space $\text{range}(A)$

● column rank = row rank = rank
Matrix Inverse

- **Nonsingular** or **invertible** matrix = square matrix with full rank
- The **inverse** $A^{-1}$ of $A$ satisfies
  \[ AA^{-1} = A^{-1}A = I \]
- Change of basis:
  \[ x = A^{-1}b = \text{solution to } Ax = b \]
  = the vector of coefficients of the expansion of $b$
  in the basis of columns of $A$