The Eigenvalue Decomposition

- Eigenvalue problem for \( m \times m \) matrix \( A \):
  \[
  Ax = \lambda x
  \]
  with eigenvalues \( \lambda \) and eigenvectors \( x \) (nonzero)

- Eigenvalue decomposition of \( A \):
  \[
  A = X\Lambda X^{-1} \quad \text{or} \quad AX = X\Lambda
  \]
  with eigenvectors as columns of \( X \) and eigenvalues on diagonal of \( \Lambda \)

- In “eigenvector coordinates”, \( A \) is diagonal:
  \[
  Ax = b \rightarrow (X^{-1}b) = \Lambda(X^{-1}x)
  \]
**Multiplicity**

- The eigenvectors corresponding to a single eigenvalue $\lambda$ (plus the zero vector) form an *eigenspace*.
- Dimension of $E_\lambda = \dim(\null(A - \lambda I)) = \text{geometric multiplicity}$ of $\lambda$.
- The characteristic polynomial of $A$ is:
  
  $$p_A(z) = \det(zI - A) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_m)$$

- $\lambda$ is eigenvalue of $A \iff p_A(\lambda) = 0$
  - Since if $\lambda$ is eigenvalue, $\lambda x - Ax = 0$. Then $\lambda I - A$ is singular, so $\det(\lambda I - A) = 0$.
- Multiplicity of a root $\lambda$ to $p_A = \text{algebraic multiplicity}$ of $\lambda$.
- Any matrix $A$ has $m$ eigenvalues, counted with algebraic multiplicity.

**Similarity Transformations**

- The map $A \mapsto X^{-1}AX$ is a *similarity transformation* of $A$.
- $A$ and $B$ are similar if there is a similarity transformation $B = X^{-1}AX$.
- $A$ and $X^{-1}AX$ have the same characteristic polynomials, eigenvalues, and multiplicities:
  - The characteristic polynomials are the same:
    
    $$p_{X^{-1}AX}(z) = \det(zI - X^{-1}AX) = \det(X^{-1}(zI - A)X)$$
    
    $$= \det(X^{-1})\det(zI - A)\det(X) = \det(zI - A) = p_A(z)$$
  - Therefore, the algebraic multiplicities are the same.
  - If $E_\lambda$ is eigenspace for $A$, then $X^{-1}E_\lambda$ is eigenspace for $X^{-1}AX$, so geometric multiplicities are the same.
**Algebraic Multiplicity \( \geq \) Geometric Multiplicity**

- Let \( n \) first columns of \( \hat{V} \) be orthonormal basis of the eigenspace for \( \lambda \)
- Extend \( \hat{V} \) to square unitary \( V \), and form

\[
B = V^* AV = \begin{bmatrix}
\lambda I & C \\
0 & D
\end{bmatrix}
\]

- Since

\[
\det(zI - B) = \det(zI - \lambda I)\det(zI - D) = (z - \lambda)^n \det(zI - D)
\]

the algebraic multiplicity of \( \lambda \) (as eigenvalue of \( B \)) is \( \geq n \)
- \( A \) and \( B \) are similar; so the same is true for \( \lambda \) of \( A \)

**Defective and Diagonalizable Matrices**

- If the algebraic multiplicity for an eigenvalue \( \geq \) its geometric multiplicity, it is a *defective eigenvalue*
- If a matrix has any defective eigenvalues, it is a *defective matrix*
- A *nondefective or diagonalizable* matrix has equal algebraic and geometric multiplicities for all eigenvalues
- The matrix \( A \) is nondefective \( \iff \) \( A = X\Lambda X^{-1} \)
  - \( (\iff) \) If \( A = X\Lambda X^{-1} \), \( A \) is similar to \( \Lambda \) and has the same eigenvalues and multiplicities. But \( \Lambda \) is diagonal and thus nondefective.
  - \( (\implies) \) Nondefective \( A \) has \( m \) linearly independent eigenvectors. Take these as the columns of \( X \), then \( A = X\Lambda X^{-1} \).
Determinant and Trace

- The trace of $A$ is $\text{tr}(A) = \sum_{j=1}^{m} a_{jj}$
- The determinant and the trace are given by the eigenvalues:
  \[
  \det(A) = \prod_{j=1}^{m} \lambda_j, \quad \text{tr}(A) = \sum_{j=1}^{m} \lambda_j
  \]

since $\det(A) = (-1)^m \det(-A) = (-1)^m p_A(0) = \prod_{j=1}^{m} \lambda_j$ and

\[
 p_A(z) = \det(zI - A) = z^m - \sum_{j=1}^{m} a_{jj} z^{m-1} + \cdots
\]

\[
 p_A(z) = (z - \lambda_1) \cdots (z - \lambda_m) = z^m - \sum_{j=1}^{m} \lambda_j z^{m-1} + \cdots
\]

Unitary Diagonalization and Schur Factorization

- A matrix $A$ is \textit{unitary diagonalizable} if, for a unitary matrix $Q$, $A = Q\Lambda Q^*$
- A hermitian matrix is unitarily diagonalizable, with real eigenvalues (because of the Schur factorization, see below)
- $A$ is unitarily diagonalizable $\iff$ $A$ is normal ($A^*A = AA^*$)
- Every square matrix $A$ has a Schur factorization $A = QTQ^*$ with unitary $Q$ and upper-triangular $T$
- Summary, Eigenvalue-Revealing Factorizations
  - Diagonalization $A = X\Lambda X^{-1}$ (nondefective $A$)
  - Unitary diagonalization $A = Q\Lambda Q^*$ (normal $A$)
  - Unitary triangularization (Schur factorization) $A = QTQ^*$ (any $A$)
Eigenvalue Algorithms

- The most obvious method is ill-conditioned: Find roots of $p_A(\lambda)$
- Instead, compute Schur factorization $A = QTQ^*$ by introducing zeros
- However, this can not be done in a finite number of steps:
  
  Any eigenvalue solver must be iterative

- To see this, consider a general polynomial of degree $m$
  \[ p(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_1z + a_0 \]
- There is no closed-form expression for the roots of $p$: (Abel, 1842)
  
  In general, the roots of polynomial equations higher than fourth degree cannot be written in terms of a finite number of operations

Eigenvalue Algorithms

- (continued) However, the roots of $p$ are the eigenvalues of the companion matrix
  \[
  A = \begin{bmatrix}
  0 & & & -a_0 \\
  1 & 0 & & -a_1 \\
  1 & 0 & & -a_2 \\
  & & \ddots & \vdots \\
  & & 0 & -a_{m-2} \\
  & & 1 & -a_{m-1}
  \end{bmatrix}
  \]
- Therefore, in general we cannot find the eigenvalues of a matrix in a finite number of steps (even in exact arithmetic)
- In practice, algorithms available converge in just a few iterations
**Schur Factorization and Diagonalization**

- Compute Schur factorization \( A = QTQ^* \) by transforming \( A \) with similarity transformations

\[
Q^*_j \cdots Q^*_2 Q^*_1 A Q^*_1 Q^*_2 \cdots Q^*_j \quad Q^*
\]

which converge to a \( T \) as \( j \to \infty \)

- Note: Real matrices might need complex Schur forms and eigenvalues (or a *real Schur factorization* with \( 2 \times 2 \) blocks on diagonal)

- For hermitian \( A \), the sequence converges to a diagonal matrix

**Two Phases of Eigenvalues Computations**

- General \( A \): First to *upper-Hessenberg* form, then to upper-triangular

\[
\begin{bmatrix}
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
A \neq A^*
\end{bmatrix}
\quad \rightarrow \quad
\begin{bmatrix}
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
H
\end{bmatrix}
\quad \rightarrow \quad
\begin{bmatrix}
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
T
\end{bmatrix}
\]

- Hermitian \( A \): First to *tridiagonal* form, then to diagonal

\[
\begin{bmatrix}
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
A \neq A^*
\end{bmatrix}
\quad \rightarrow \quad
\begin{bmatrix}
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
T
\end{bmatrix}
\quad \rightarrow \quad
\begin{bmatrix}
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
D
\end{bmatrix}
\]