Lecture 23
Arnoldi/Lanczos Iterations

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Introduction to Numerical Methods

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The Arnoldi Iteration

- The Arnoldi process reduces a general, nonsymmetric $A$ to Hessenberg form by similarity transforms: $A = QHQ^*$
- Allows for reduced factorizations by a Gram-Schmidt-style iteration instead of Householder reflections
- Let $Q_n$ be the $m \times n$ matrix with the first $n$ columns of $Q$, and consider the first $n$ columns of $AQ = QH$, or $AQ_n = Q_{n+1}\tilde{H}_n$:

\[
\begin{bmatrix}
A \\
q_1 & \cdots & q_n
\end{bmatrix}
\begin{bmatrix}
q_1 \\
\cdots \\
q_n
\end{bmatrix}
= 
\begin{bmatrix}
q_1 \\
\cdots \\
q_{n+1}
\end{bmatrix}
\begin{bmatrix}
h_{11} & \cdots & h_{1n} \\
h_{21} \\
\vdots \\
h_{n+1,n}
\end{bmatrix}
\]
The Arnoldi Algorithm

- The \( n \)th column of \( AQ_n = Q_{n+1} \tilde{H}_n \) gives

\[
AQ_n = h_{1n}q_1 + \cdots + h_{nn}q_n + h_{n+1,n}q_{n+1}
\]

which can be used to compute \( q_{n+1} \) similarly to modified Gram-Schmidt:

**Algorithm: Arnoldi Iteration**

\[
b = \text{arbitrary}, \quad q_1 = b / \|b\| \\
\text{for } n = 1, 2, 3, \ldots \\
\quad v = AQ_n \\
\quad \text{for } j = 1 \text{ to } n \\
\quad \quad h_{jn} = q_j^*v \\
\quad \quad v = v - h_{jn}q_j \\
\quad h_{n+1,n} = \|v\| \\
\quad q_{n+1} = v / h_{n+1,n}
\]
The vectors $q_j$ from Arnoldi are orthonormal bases of the successive Krylov subspaces:

$$K_n = \langle b, Ab, \ldots, A^{n-1}b \rangle = \langle q_1, q_2, \ldots, q_n \rangle \subseteq \mathbb{C}^m$$

$Q_n$ is the reduced QR factorization $K_n = Q_nR_n$ of the Krylov matrix:

$$K_n = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}$$

The projection of $A$ onto this space gives $n \times n$ Hessenberg matrix $H_n = Q_n^*AQ_n$, whose eigenvalues may be good approximations of $A$'s
Symmetric Matrices and the Lanczos Iteration

- For symmetric $A$, $H_n$ reduces to tridiagonal $T_n$, and $q_{n+1}$ can be computed by a three-term recurrence:

$$Aq_n = \beta_{n-1}q_{n-1} + \alpha_n q_n + \beta_n q_{n+1}$$

### Algorithm: Lanczos Iteration

- $\beta_0 = 0$, $q_0 = 0$, $b =$ arbitrary, $q_1 = b / \|b\|$
- **for** $n = 1, 2, 3, \ldots$
  - $v = Aq_n$
  - $\alpha_n = q_n^T v$
  - $v = v - \beta_{n-1}q_{n-1} - \alpha_n q_n$
  - $\beta_n = \|v\|$
  - $q_{n+1} = v / \beta_n$