The Arnoldi Iteration

- The Arnoldi process reduces a general, nonsymmetric $A$ to Hessenberg form by similarity transforms: $A = QHQ^*$

- Allows for reduced factorizations by a Gram-Schmidt-style iteration instead of Householder reflections

- Let $Q_n$ be the $m \times n$ matrix with the first $n$ columns of $Q$, and consider the first $n$ columns of $AQ = QH$, or $AQ_n = Q_{n+1} \tilde{H}_n$:

$$A \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} = \begin{bmatrix} q_1 & \cdots & q_{n+1} \end{bmatrix} \begin{bmatrix} h_{11} & \cdots & h_{1n} \\ h_{21} & \ddots & \vdots \\ \vdots & \ddots & h_{n+1,n} \end{bmatrix}.$$
The Arnoldi Algorithm

- The $n$th column of $AQ_n = Q_{n+1}\tilde{H}_n$ gives

$$AQ_n = h_{1n}q_1 + \cdots + h_{nn}q_n + h_{n+1,n}q_{n+1}$$

which can be used to compute $q_{n+1}$ similarly to modified Gram-Schmidt:

```
Algorithm: Arnoldi Iteration
b = arbitrary, q_1 = b/\|b\|
for n = 1, 2, 3, \ldots
    v = Aq_n
    for j = 1 to n
        h_{jn} = q_j^*v
        v = v - h_{jn}q_j
    h_{n+1,n} = \|v\|
    q_{n+1} = v/h_{n+1,n}
```

QR Factorization of Krylov Matrix

- The vectors $q_j$ from Arnoldi are orthonormal bases of the successive Krylov subspaces:

$$\mathcal{K}_n = \langle b, Ab, \ldots, A^{n-1}b \rangle = \langle q_1, q_2, \ldots, q_n \rangle \subset \mathbb{C}^m$$

- $Q_n$ is the reduced QR factorization $K_n = Q_nR_n$ of the Krylov matrix:

$$K_n = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}$$

- The projection of $A$ onto this space gives $n \times n$ Hessenberg matrix $H_n = Q_n^*AQ_n$, whose eigenvalues may be good approximations of $A$'s
Symmetric Matrices and the Lanczos Iteration

- For symmetric $A$, $H_n$ reduces to tridiagonal $T_n$, and $q_{n+1}$ can be computed by a three-term recurrence:

$$Aq_n = \beta_{n-1}q_{n-1} + \alpha_nq_n + \beta_nq_{n+1}$$

**Algorithm: Lanczos Iteration**

\[
\begin{align*}
\beta_0 &= 0, \quad q_0 = 0, \quad b = \text{arbitrary}, \quad q_1 = b/\|b\| \\
\text{for } n = 1, 2, 3, \ldots \\
&\quad \v = Aq_n \\
&\quad \alpha_n = q_n^Tv \\
&\quad v = v - \beta_{n-1}q_{n-1} - \alpha_nq_n \\
&\quad \beta_n = \|v\| \\
&\quad q_{n+1} = v/\beta_n
\end{align*}
\]