The SVD - The Main Idea

- Motivation:

The image of the unit sphere under any $m \times n$ matrix is a hyperellipse
The SVD - Brief Description

- Suppose (for the moment) that \( A \) is \( m \times n \) with \( m \geq n \) and full rank \( n \)
- Choose orthonormal bases

\[
v_1, \ldots, v_n \text{ for the row space} \]
\[
u_1, \ldots, u_n \text{ for the column space} \]

such that \( Av_i \) is in the direction of \( u_i \):

\[
Av_i = \sigma_i u_i
\]

- In MATLAB: `eigshow`
- The singular values \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0 \)

The SVD - Brief Description

- In Matrix form, \( Av_i = \sigma_i u_i \) becomes

\[
AV = \hat{U} \hat{\Sigma}, \quad \text{that is,} \quad A = \hat{U} \hat{\Sigma} V^* 
\]

where \( \hat{\Sigma} = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \)

- This is the reduced singular value decomposition

- Add orthogonal extension to \( \hat{U} \) and add rows to \( \hat{\Sigma} \) to obtain the full singular value decomposition

\[
A = U \Sigma V^*
\]
The Full Singular Value Decomposition

- Let $A$ be an $m \times n$ matrix. The singular value decomposition of $A$ is the factorization $A = U \Sigma V^*$ where

  $U$ is $m \times m$ unitary (the left singular vectors of $A$)
  $V$ is $n \times n$ unitary (the right singular vectors of $A$)
  $\Sigma$ is $m \times n$ diagonal (the singular values of $A$)

\[
A = \begin{bmatrix}
U & \Sigma & V^* \\
\end{bmatrix}
\]

The Reduced Singular Value Decomposition

- A more compact representation is the Reduced SVD, for $m \geq n$:

  \[
  A = \hat{U} \hat{\Sigma} V^*
  \]

  where

  $\hat{U}$ is $m \times n$, $V$ is $n \times n$, and $\Sigma$ is $n \times n$

\[
A = \begin{bmatrix}
\hat{U} & \hat{\Sigma} & V^* \\
\end{bmatrix}
\]
The SVD and The Eigenvalue Decomposition

- The eigenvalue decomposition $A = X \Lambda X^{-1}$
  - uses the same basis $X$ for row and column space, but the SVD uses two different bases $V, U$
  - generally does not use an orthonormal basis, but the SVD does
  - is only defined for square matrices, but the SVD exists for all matrices
- For symmetric positive definite matrices $A$, the eigenvalue decomposition and the SVD are equal

Matrix Properties

1. The rank of $A$ is $r$, the number of nonzero singular values
2. range$(A) = \langle u_1, \ldots, u_r \rangle$ and null$(A) = \langle v_{r+1}, \ldots, v_n \rangle$
3. $\|A\|_2 = \sigma_1$ and $\|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_r^2}$
4. Nonzero eigenvalues of $A^*A$ are nonzero $\sigma_i^2$, eigenvectors are $v_i$
   Nonzero eigenvalues of $AA^*$ are nonzero $\sigma_i^2$, eigenvectors are $u_i$
5. If $A = A^*$, $\sigma_i = |\lambda_i|$ where $\lambda_i$ are eigenvalues of $A$
6. For square $A$, $|\det(A)| = \prod_{i=1}^m \sigma_i$
Existence and Uniqueness

- Every matrix has a singular value decomposition
- The singular values $\sigma_j$ are uniquely determined
- If $A$ square and $\sigma_j$ distinct, left/right singular vectors $u_j, v_j$ are uniquely determined up to complex signs
- Proof. Textbook / Black board

Low-Rank Approximations

- The SVD can be written as a sum of rank-one matrices
  \[ A = \sum_{j=1}^{r} \sigma_j u_j v_j^* \]
- The best rank $\nu$ approximation of $A$ in the 2-norm is
  \[ A_\nu = \sum_{j=1}^{\nu} \sigma_j u_j v_j^* \]
  with $\| A - A_\nu \|_2 = \sigma_{\nu+1}$
- Also true in the Frobenius norm, with $\| A - A_\nu \|_2 = \sqrt{\sigma_{\nu+1}^2 + \cdots + \sigma_r^2}$
Applications of the SVD

- Calculation of matrix properties:
  - Rank of matrix (counting $\sigma_j$’s $> \text{tolerance}$)
  - Bases for range and nullspace (in $U$ and $V$)
  - Induced matrix norm $\| \cdot \|_2 (= \sigma_1)$

- Low-rank approximations (optimal in $\| \cdot \|_2$ and $\| \cdot \|_F$)

- Least squares fitting (more later, another option is $QR$)

- Signal and image processing
  - Compression (see next slide)
  - Noise removal (noise tends to have low $\sigma_j$)

Application: Image Compression

- View $m \times n$ image as a (real) matrix $A$, find best rank $\nu$ approx. by SVD

- Storage $\nu(m + n)$ instead of $mn$