Projectors

- A projector is a square matrix $P$ that satisfies
  \[ P^2 = P \]

- Not necessarily an orthogonal projector (more later)

- If $v \in \text{range}(P)$, then $Pv = v$
  - Since with $v = Px$,
    \[ Pv = P^2x = Px = v \]

- Projection along the line $Pv - v \in \text{null}(P)$
  - Since $P(Pv - v) = P^2v - Pv = 0$
Complementary Projectors

- The matrix $I - P$ is the *complementary projector* to $P$
- $I - P$ projects on the nullspace of $P$:
  - If $Pv = 0$, then $(I - P)v = v$, so $\text{null}(P) \subseteq \text{range}(I - P)$
  - But for any $v$, $(I - P)v = v - Pv \in \text{null}(P)$, so $\text{range}(I - P) \subseteq \text{null}(P)$
  - Therefore
    \[
    \text{range}(I - P) = \text{null}(P)
    \]
    and
    \[
    \text{null}(I - P) = \text{range}(P)
    \]

Complementary Subspaces

- For a projector $P$,
  \[
  \text{null}(I - P) \cap \text{null}(P) = \{0\}
  \]
  or
  \[
  \text{range}(P) \cap \text{null}(P) = \{0\}
  \]
- A projector separates $\mathbb{C}^m$ into two spaces $S_1, S_2$, with $\text{range}(P) = S_1$ and $\text{null}(P) = S_2$
- $P$ is the projector *onto* $S_1$ *along* $S_2$
Orthogonal Projectors

- An *orthogonal projector* projects onto $S_1$ along $S_2$, with $S_1, S_2$ orthogonal
- A projector $P$ is orthogonal $\iff P = P^*$
- *Proof*. Textbook / Black board

\[ \text{range}(P) \]

\[ Pv - v \]

Projection with Orthonormal Basis

- Reduced SVD gives projector for orthonormal columns $\hat{Q}$:
  \[ P = \hat{Q}\hat{Q}^* \]
- Complement $I - \hat{Q}\hat{Q}^*$ also orthogonal, projects onto space orthogonal to $\text{range}(\hat{Q})$
- Special case 1: Rank-1 Orthogonal Projector (gives component in direction $q$)
  \[ P_q = qq^* \]
- Special case 2: Rank $m - 1$ Orthogonal Projector (eliminates component in direction $q$)
  \[ P_{\perp q} = I - qq^* \]
Projection with Arbitrary Basis

- Project $v$ to $y \in \text{range}(A)$. Then
  
  \[
  y - v \perp \text{range}(A), \text{ or } a_j^*(y - v) = 0, \forall j
  \]

- Set $y = Ax$:
  
  \[
  a_j^*(Ax - v) = 0, \forall j \iff A^*(Ax - v) = 0 \iff A^*Ax = A^*v
  \]

- $A^*A$ is nonsingular, so
  
  \[
  x = (A^*A)^{-1}A^*v
  \]

- Finally, we are interested in the projection $y = Ax = A(A^*A)^{-1}A^*v$, giving the orthogonal projector
  
  \[
  P = A(A^*A)^{-1}A^*
  \]

The QR Factorization - Main Idea

- Find orthonormal vectors that span the successive spaces spanned by the columns of $A$:
  
  \[
  \langle a_1 \rangle \subseteq \langle a_1, a_2 \rangle \subseteq \langle a_1, a_2, a_3 \rangle \subseteq \ldots
  \]

- This means that (for full rank $A$),
  
  \[
  \langle q_1, q_2, \ldots, q_j \rangle = \langle a_1, a_2, \ldots, a_j \rangle, \text{ for } j = 1, \ldots, n
  \]
The QR Factorization - Matrix Form

- In matrix form, \(\langle q_1, q_2, \ldots, q_j \rangle = \langle a_1, a_2, \ldots, a_j \rangle\) becomes

\[
\begin{bmatrix}
a_1 & a_2 & \cdots & a_n
\end{bmatrix}
= \begin{bmatrix}
q_1 & q_2 & \cdots & q_n
\end{bmatrix}
\begin{bmatrix}
r_{11} & r_{12} & \cdots & r_{1n}

r_{22} & & & \\
& \ddots & & \\
& & & \ddots \\
r_{nn} & & & \\
\end{bmatrix}
\]

or

\[
A = \hat{Q}\hat{R}
\]

- This is the reduced QR factorization
- Add orthogonal extension to \(\hat{Q}\) and add rows to \(\hat{R}\) to obtain the full QR factorization

The Full QR Factorization

- Let \(A\) be an \(m \times n\) matrix. The full QR factorization of \(A\) is the factorization \(A = QR\), where

\[
Q \text{ is } m \times m \text{ unitary}
\]

\[
R \text{ is } m \times n \text{ upper-triangular}
\]
The Reduced QR Factorization

- A more compact representation is the Reduced QR Factorization $A = \hat{Q} \hat{R}$, where (for $m \geq n$)

\[
\hat{Q} \text{ is } m \times n \text{ and } \hat{R} \text{ is } m \times n
\]

Gram-Schmidt Orthogonalization

- Find new $q_j$ orthogonal to $q_1, \ldots, q_{j-1}$ by subtracting components along previous vectors

\[
v_j = a_j - (q_1^* a_j)q_1 - (q_2^* a_j)q_2 - \cdots - (q_{j-1}^* a_j)q_{j-1}
\]

- Normalize to get $q_j = v_j / \|v_j\|

- We then obtain a reduced QR factorization $A = \hat{Q} \hat{R}$, with

\[
r_{ij} = q_i^* a_j, \quad (i \neq j)
\]

and

\[
|r_{jj}| = \|a_j - \sum_{i=1}^{j-1} r_{ij} q_i\|_2
\]
Classical Gram-Schmidt

- Straight-forward application of Gram-Schmidt orthogonalization
- Numerically unstable

### Algorithm: Classical Gram-Schmidt

```
for j = 1 to n
    v_j = a_j
    for i = 1 to j - 1
        r_{ij} = q_i^* a_j
        v_j = v_j - r_{ij} q_i
    r_{jj} = \|v_j\|_2
    q_j = v_j / r_{jj}
```

Existence and Uniqueness

- Every \( A \in \mathbb{C}^{m \times n} (m \geq n) \) has a full QR factorization and a reduced QR factorization

  **Proof.** For full rank \( A \), Gram-Schmidt proves existence of \( A = \hat{Q} \hat{R} \).
  Otherwise, when \( v_j = 0 \) choose arbitrary vector orthogonal to previous \( q_i \).
  For full QR, add orthogonal extension to \( \hat{Q} \) and zero rows to \( \hat{R} \).

- Each \( A \in \mathbb{C}^{m \times n} (m \geq n) \) of full rank has unique \( A = \hat{Q} \hat{R} \) with \( r_{jj} > 0 \)

  **Proof.** Again Gram-Schmidt, \( r_{jj} > 0 \) determines the sign