Projectors

A projector is a square matrix $P$ that satisfies

$$P^2 = P$$

Not necessarily an orthogonal projector (more later)

If $v \in \text{range}(P)$, then $Pv = v$

- Since with $v = Px$, $Pv = P^2x = Px = v$

Projection along the line $Pv - v \in \text{null}(P)$

- Since $P(Pv - v) = P^2v - Pv = 0$

Complementary Projectors

The matrix $I - P$ is the complementary projector to $P$

$I - P$ projects on the nullspace of $P$:
- If $Pv = 0$, then $(I - P)v = v$, so $\text{null}(P) \subseteq \text{range}(I - P)$
- But for any $v$, $(I - P)v = v - Pv \in \text{null}(P)$, so $\text{range}(I - P) \subseteq \text{null}(P)$
- Therefore

$$\text{range}(I - P) = \text{null}(P)$$

and

$$\text{null}(I - P) = \text{range}(P)$$

Complementary Subspaces

For a projector $P$,

$$\text{null}(I - P) \cap \text{null}(P) = \{0\}$$

or

$$\text{range}(P) \cap \text{null}(P) = \{0\}$$

A projector separates $\mathbb{C}^m$ into two spaces $S_1$, $S_2$, with $\text{range}(P) = S_1$ and $\text{null}(P) = S_2$

$P$ is the projector onto $S_1$ along $S_2$

Orthogonal Projectors

- An orthogonal projector projects onto $S_1$ along $S_2$, with $S_1$, $S_2$ orthogonal
- A projector $P$ is orthogonal $\iff P = P^*$
- Proof. Textbook / Black board

Projection with Orthonormal Basis

- Reduced SVD gives projector for orthonormal columns $\hat{Q}$:

$$P = \hat{Q}\hat{Q}^*$$

- Complement $I - \hat{Q}\hat{Q}^*$ also orthogonal, projects onto space orthogonal to $\text{range}(\hat{Q})$
- Special case 1: Rank-1 Orthogonal Projector (gives component in direction $q$)

$$P_q = qq^*$$

- Special case 2: Rank $m - 1$ Orthogonal Projector (eliminates component in direction $q$)

$$P_{\perp q} = I - qq^*$$
Projection with Arbitrary Basis

- Project $v$ to $y \in \text{range}(A)$. Then 
  $$y - v \perp \text{range}(A), \text{ or } a_j^*(y - v) = 0, \forall j$$
- Set $y = Ax$: 
  $$a_j^*(Ax - v) = 0, \forall j \iff A^*(Ax - v) = 0 \iff A^*Ax = A^*v$$
- $A^*A$ is nonsingular, so 
  $$x = (A^*A)^{-1}A^*v$$
- Finally, we are interested in the projection $y = Ax = A(A^*A)^{-1}A^*v$, giving the orthogonal projector 
  $$P = A(A^*A)^{-1}A^*$$

The QR Factorization - Main Idea

- Find orthonormal vectors that span the successive spaces spanned by the columns of $A$: 
  $$\langle a_1 \rangle \subseteq \langle a_1, a_2 \rangle \subseteq \langle a_1, a_2, a_3 \rangle \subseteq \ldots$$
- This means that (for full rank $A$), 
  $$\langle q_1, q_2, \ldots, q_j \rangle = \langle a_1, a_2, \ldots, a_j \rangle, \text{ for } j = 1, \ldots, n$$

The QR Factorization - Matrix Form

- In matrix form, $\langle q_1, q_2, \ldots, q_j \rangle = \langle a_1, a_2, \ldots, a_j \rangle$ becomes 
  $$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{bmatrix}$$
  or 
  $$A = \tilde{Q}\tilde{R}$$
- This is the reduced QR factorization 
- Add orthogonal extension to $\tilde{Q}$ and add rows to $\tilde{R}$ to obtain the full QR factorization

The Full QR Factorization

- Let $A$ be an $m \times n$ matrix. The full QR factorization of $A$ is the factorization $A = QR$, where 
  $$Q \text{ is } m \times m \text{ unitary}$$
  $$R \text{ is } m \times n \text{ upper-triangular}$$
  $$A = \begin{bmatrix} Q & R \end{bmatrix}$$

The Reduced QR Factorization

- A more compact representation is the Reduced QR Factorization $A = \tilde{Q}\tilde{R}$, where (for $m \geq n$) 
  $$\tilde{Q} \text{ is } m \times n \text{ and } \tilde{R} \text{ is } m \times n$$
  $$A = \begin{bmatrix} \tilde{Q} & \tilde{R} \end{bmatrix}$$

Gram-Schmidt Orthogonalization

- Find new $q_j$ orthogonal to $q_1, \ldots, q_{j-1}$ by subtracting components along previous vectors 
  $$v_j = a_j - (q_1^*a_j)q_1 - (q_2^*a_j)q_2 - \cdots - (q_{j-1}^*a_j)q_{j-1}$$
- Normalize to get $q_j = v_j / \|v_j\|$ 
- We then obtain a reduced QR factorization $A = \hat{Q}\hat{R}$, with 
  $$r_{ij} = q_i^*a_j, \quad (i \neq j)$$
  and 
  $$|r_{jj}| = \|a_j - \sum_{i=1}^{j-1} r_{ij}q_i\|_2$$
Classical Gram-Schmidt

- Straight-forward application of Gram-Schmidt orthogonalization
- Numerically unstable

Algorithm: Classical Gram-Schmidt

for $j = 1$ to $n$
  
  $v_j = a_j$

  for $i = 1$ to $j - 1$
    
    $r_{ij} = q_i^* a_j$
    
    $v_j = v_j - r_{ij} q_i$

  $r_{jj} = \|v_j\|_2$

  $q_j = v_j / r_{jj}$

Existence and Uniqueness

- Every $A \in \mathbb{C}^{m \times n}$ ($m \geq n$) has a full QR factorization and a reduced QR factorization

  \textbf{Proof.} For full rank $A$, Gram-Schmidt proves existence of $A = \hat{Q}\hat{R}$.

  Otherwise, when $v_j = 0$ choose arbitrary vector orthogonal to previous $q_i$.

  For full QR, add orthogonal extension to $\hat{Q}$ and zero rows to $\hat{R}$.

- Each $A \in \mathbb{C}^{m \times n}$ ($m \geq n$) of full rank has unique $A = \hat{Q}\hat{R}$ with $r_{jj} > 0$

  \textbf{Proof.} Again Gram-Schmidt, $r_{jj} > 0$ determines the sign